VECTOR STOCHASTIC INTEGRALS IN THE FUNDAMENTAL THEOREM OF ASSET PRICING

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1 Introduction

The *Fundamental Theorem of Asset Pricing* asserts that for a d-dimensional semimartingale X, the absence of specifically defined arbitrage opportunities implies the existence of an equivalent measure such that X is a *martingale transform* with respect to this measure. Here, the definition of arbitrage employs *vector stochastic integrals* $H \bullet X$ where H belongs to the set of X-integrable predictable processes. The question arises whether one may deal here with *componentwise integrals* or only with locally bounded integrands. It turns out that the answer to this question depends on the definition of the absence of arbitrage. That is, the condition $(\widetilde{NA})$ for locally bounded integrands implies the existence of an equivalent measure for which X is a martingale transform. It is proved in section 3 of this paper. On the other hand, in section 4 we present an example of a two-dimensional semimartingale satisfying the condition $(\overline{NA})$ for componentwise integrals and possessing no equivalent measure such that X is a martingale transform with respect to this measure. This example illustrates the importance of vector stochastic integrals in the mathematical finance.

2 Preliminaries

Let $X = (X_t^1, \dots, X_t^d)_{t\ge 0}$ be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$.

Definition 2.1 An $\mathcal{F}_t$-predictable process $H = (H_t^1, \dots, H_t^d)$ is *X-integrable* if there exists a decomposition $X = A + M$ into a sum of a finite-variation process and a local martingale such that

$$\int_0^t \Big|\sum_{i=1}^d H_s^i a_s^i\Big| dC_s < \infty \quad \text{a.s.} \quad \forall t > 0,$$

$$\Big(\int_0^t \Big(\sum_{i,j=1}^d H_s^i \pi_s^{ij} H_s^j\Big) dC_s\Big)^{1/2} \in \mathcal{A}_{loc}$$

where $\mathcal{F}_t$-adapted processes a_t^i, π_t^{ij} and an increasing process C_t satisfy the following conditions:

$$A_t^i = \int_0^t a_s^i dC_s \quad (i = 1, \dots, d),$$
$$[M^i, M^j]_t = \int_0^t \pi_s^{ij} dC_s \quad (i, j = 1, \dots, d).$$

The set of all X-integrable predictable processes is denoted by $L(X)$ whereas $(H \bullet X)_t$ designates the *vector stochastic integral* $\int_0^t H_s dX_s$. For the definition of a vector stochastic integral, we refer to [7] and [11].

Remarks. **1)** A vector stochastic integral is a one-dimensional semimartingale defined up to indistinguishability.

2) In the mathematical finance, the process X is interpreted as a discounted price process of d assets on the securities market. The process H describes the self-financing portfolio with the capital process $H \bullet X$.

A process $H = (H_t^1, \dots, H_t^d)$ is said to be *componentwise X-integrable* if $H^i \in L(X^i)$ for $i = 1, \dots, d$. For instance, any locally bounded predictable process satisfies this condition. It is well known that for any componentwise X-integrable process H, we have $H \in L(X)$ and

$$H \bullet X = \sum_{i=1}^{d} H^i \bullet X^i$$

where $H^i \bullet X^i$ are one-dimensional stochastic integrals. The following example shows that the converse is false.

Example 2.2 Let Y_t be a one-dimensional local martingale and K_t be a predictable process such that $K \notin L(Y)$. If we set $X_t = (Y_t, Y_t)$, $H_t = (K_t, -K_t)$, $A_t = 0$ and $M_t = X_t$, then H satisfies the integrability assumption from Definition 2.1 since

$$\sum_{i,j=1}^{2} H_t^i \pi_t^{ij} H_t^j = 0$$

due to the equality $\pi_t^{11} = \pi_t^{12} = \pi_t^{21} = \pi_t^{22}$. Thus, $H \in L(X)$ and $H \bullet X = 0$ while $H^i \notin L(X^i)$ for $i = 1, 2$. $\square$

Let us now recall two important theorems on vector stochastic integration.

Theorem 2.3 (i) If X is a d-dimensional semimartingale and $H_i \in L(X)$ $(i = 1, 2)$, then $\alpha H_1 + \beta H_2 \in L(X)$ and

$$(\alpha H_1 + \beta H_2) \bullet X = \alpha(H_1 \bullet X) + \beta(H_2 \bullet X).$$

(ii) If X_1, X_2 are d-dimensional semimartingales and $H \in L(X_i)$ $(i = 1, 2)$, then $H \in L(\alpha X_1 + \beta X_2)$ and

$$H \bullet (\alpha X_1 + \beta X_2) = \alpha(H \bullet X_1) + \beta(H \bullet X_2).$$

Theorem 2.4 Let X be a d-dimensional semimartingale and suppose that $H \in L(X)$. Then, for any one-dimensional process K, we have: $K \in L(H \bullet X) \Leftrightarrow (KH) \in L(X)$, in which case $K \bullet (H \bullet X) = (KH) \bullet X$.

We now turn to various definitions for the absence of arbitrage.

Definition 2.5 **1)** An integrand $H = (H_t^1, \ldots, H_t^d)$ realizes $(A)_V$ (*Arbitrage*) on X where V stands for "vector", if

- (i) $H \in L(X)$;
- (ii) there exists $a \geq 0$ such that $H \bullet X \geq -a$ a.s., i.e., $P\big((H \bullet X)_t \geq -a\big) = 1$ for each $t \geq 0$; due to right-continuity of $H \bullet X$, it is equivalent to the condition: $P\big((H \bullet X)_t \geq -a \quad \forall t \geq 0\big) = 1$;
- (iii) almost surely there exists the limit $(H \bullet X)_\infty = \lim\limits_{t \to \infty} (H \bullet X)_t$;
- (iv) $(H \bullet X)_\infty \geq 0$ a.s. and $P\big((H \bullet X)_\infty > 0\big) > 0$.

2) An integrand H realizes $(A)_C$ on X where C stands for "componentwise", if the condition (i) is replaced by:

- $(i)'$ $H^i \in L(X^i)$ for $i = 1, \ldots, d$.

3) An integrand H realizes $(A)_B$ on X where B stands for "bounded", if H is locally bounded and all the conditions stated above are fulfilled.
4) By *No Arbitrage* properties $(NA)_V$,$(NA)_C$ and $(NA)_B$, we mean the absence of corresponding integrands for X.

Definition 2.6 **1)** A sequence of integrands $H_n = \big((H_n^1)_t, \ldots, (H_n^d)_t\big)$ realizes $(\overline{A})_V$ on X if

- (i) $H_n \in L(X)$ for $n \in I\!N$;
- (ii) for each $n \in I\!N$, there exists $a_n \geq 0$ such that $H_n \bullet X \geq -a_n$ a.s.;
- (iii) almost surely there exists the limit $(H_n \bullet X)_\infty = \lim\limits_{t \to \infty} (H_n \bullet X)_t$ for each $n \in I\!N$;
- (iv) $(H_n \bullet X)_\infty \geq -1/n$ a.s. and there exist $\delta_1 > 0, \delta_2 > 0$ such that $P\big((H_n \bullet X)_\infty > \delta_1\big) > \delta_2$ for each $n \in I\!N$.

2) A sequence of integrands H_n realizes $(\overline{A})_C$ on X if the condition (i) is replaced by:

- $(i)'$ $H_n^i \in L(X^i)$ for $i = 1, \ldots, d, n \in I\!N$.

3) A sequence of integrands H_n realizes $(\overline{A})_B$ on X if H_n is locally bounded for $n \in I\!N$ and all the conditions stated above are fulfilled.
4) By the properties $(\overline{NA})_V$,$(\overline{NA})_C$ and $(\overline{NA})_B$, we mean the absence of corresponding integrands for X.

Definition 2.7 The properties $(\widetilde{NA})_V$, $(\widetilde{NA})_C$ and $(\widetilde{NA})_B$ are defined similarly to $(\overline{NA})$ properties; namely, the condition (iv) should be replaced by:

$(iv)'$ $(H_n \bullet X)_\infty \geq -b$ a.s. for some $b \geq 0$, $\lim_{n\to\infty} P\big((H_n \bullet X)_\infty < -\delta\big) = 0$ for each $\delta > 0$ and there exist $\delta_1 > 0, \delta_2 > 0$ such that $P\big((H_n \bullet X)_\infty > \delta_1\big) > \delta_2$ for each $n \in I\!N$.

Remarks. **1)** The following implications are obvious:

$$\begin{array}{ccccc}
(\widetilde{NA})_V & \Longrightarrow & (\widetilde{NA})_C & \Longrightarrow & (\widetilde{NA})_B \\
\Downarrow & & \Downarrow & & \Downarrow \\
(\overline{NA})_V & \Longrightarrow & (\overline{NA})_C & \Longrightarrow & (\overline{NA})_B \\
\Downarrow & & \Downarrow & & \Downarrow \\
(NA)_V & \Longrightarrow & (NA)_C & \Longrightarrow & (NA)_B
\end{array}$$

2) The notions $(\overline{NA})_V$ and $(\widetilde{NA})_V$ were introduced by Delbaen and Schachermayer in [2] and designated by *No Free Lunch with Vanising Risk* $(NFLVR)$ and *No Free Lunch with Bounded Risk* $(NFLBR)$. We shall now cite the definitions of these notions from [2].

Let K_0 be a subset of $L^0 = L^0(\Omega, \mathcal{F}, P)$, defined as

$$K_0 = \{(H{\bullet}X)_\infty \mid H \in L(X),\ \exists a : H{\bullet}X \geq -a \text{ a.s. and } (H{\bullet}X)_\infty = \lim_{t\to\infty}(H{\bullet}X)_t \text{ a.s.}\}.$$

Set

$$C_0 = K_0 - L^0_+, \quad C = C_0 \cap L^\infty.$$

By $\overline{C}$ and $\widetilde{C}$, we denote the norm closure and weak* sequential closure in L^∞. The properties $(NFLVR)$ and $(NFLBR)$ mean that $\overline{C} \cap L^\infty_+ = \{0\}$ and $\widetilde{C} \cap L^\infty_+ = \{0\}$. These are equivalent to $(\overline{NA})_V$ and $(\widetilde{NA})_V$, respectively.

The following notion was introduced in [5] under the name "semimartingales de la classe $(\sum_m)$".

Definition 2.8 A d-dimensional semimartingale X is called a *martingale transform* if there exist a d-dimensional local martingale M and a componentwise M-integrable process H such that $X^i = H^i \bullet M^i$ for $i = 1, \ldots, d$.

Remarks. **1)** For a semimartingale X, there is equivalence between:

(a) X is a martingale transform.

(b) There exist $M \in \mathcal{H}^1$ and a positive process φ_t such that $\varphi \in L(M^i)$ and $X^i = \varphi \bullet M^i$ $(i = 1, \ldots, d)$.

2) In the discrete-time case, the class of martingale transforms coincides with the class of local martingales (see [9]). On the contrary, in the continuous-time case, the situation is different. Émery (see [5]) presented an example of a martingale transform which is not a local martingale. However, from the theorem proved by Ansel and Stricker (see [1]), it follows that any locally bounded martingale transform is a local martingale.

Now, we formulate the *Fundamental Theorem of Asset Pricing* (for the proof, see [3]).

Theorem 2.9 There is equivalence between:

(a) X satisfies $(\widetilde{NA})_V$.

(b) X satisfies $(\overline{NA})_V$.

(c) There exists a probability measure $Q \sim P$ such that X is a martingale transform with respect to Q.

Finally, we recall that for one-dimensional semimartingales X and Y, the *Émery distance* between them equals

$$D(X,Y) = \sup_{|H| \le 1} \Big(\sum_{n=1}^{\infty} 2^{-n} E\big[\min\big\{|(H \bullet X)_n|; 1\big\}\big] \Big)$$

where sup is taken over the set of all predictable processes H bounded by 1 (this metric was introduced by Émery in [4]). The corresponding topology is called a *semimartingale* or *Émery topology.* Convergence in this topology is stronger than uniform convergence on compact time intervals in probability.

3 Positive Results

In this section, we prove that any of the conditions $(\widetilde{NA})_B$, $(\widetilde{NA})_C$ implies the existence of an equivalent measure for which X is a martingale transform.

Proposition 3.1 Together with $(NA)_V$, the condition $(\overline{NA})_B$ implies $(\overline{NA})_V$.

Proof. Suppose that X satisfies $(\overline{NA})_B$ and $(NA)_V$ but does not satisfy $(\overline{NA})_V$. Let H_n be a sequence of integrands realizing $(\overline{A})_V$ on X. We first prove that

$$H_n \bullet X \ge -\frac{1}{n} \quad \text{a.s.,} \quad n \in I\!N. \tag{3.2}$$

If this condition is violated, then for some $n_0 \in N$ and $t_0 > 0$, we have

$$P\Big((H_{n_0} \bullet X)_{t_0} < -\frac{1}{n_0}\Big) > 0.$$

By letting

$$A = \Big\{(H_{n_0} \bullet X)_{t_0} < -\frac{1}{n_0}\Big\},$$
$$K = H_{n_0} \cdot I\big(A \times (t_0, \infty)\big),$$

we obtain an integrand realizing $(A)_V$ on X.

For

$$H_{nm} = H_n \cdot I(|H_n| \le m),$$

we have that $H_{nm} \bullet X$ converge to $H_n \bullet X$ in the Émery topology as $m \to \infty$ (see [10]) and therefore

$$P\big(\sup_{t \in [0,T]} |(H_{nm} \bullet X)_t - (H_n \bullet X)_t| > \delta\big) \longrightarrow 0, \quad m \to \infty$$

for $T \geq 0, \delta > 0, n \in I\!N$. The definition of $(\overline{A})_V$ gives constants $\delta_1 > 0, \delta_2 > 0$ and a sequence T_n such that

$$P\Big((H_n \bullet X)_{T_n} > \delta_1\Big) > \delta_2, \quad n \in I\!N. \tag{3.3}$$

For each $n \in I\!N$, we can find $m(n)$ satisfying the condition:

$$P\Big(\sup_{t\in[0,T_n]} |(H_{n\,m(n)} \bullet X)_t - (H_n \bullet X)_t| \geq \frac{1}{n}\Big) < \frac{1}{n}. \tag{3.4}$$

If we set

$$K_n^0 = H_{n\,m(n)} \cdot I\big([0, T_n]\big),$$
$$\tau_n' = \inf\big\{t \geq 0 : (K_n^0 \bullet X)_t \leq (H_n \bullet X)_t - \frac{2}{n}\big\},$$
$$\tau_n = \tau_n' \wedge T_n,$$

then the properties of stochastic integrals imply

$$\big|\Delta(K_n^0 \bullet X)_{\tau_n}\big| = \big|(K_n^0)_{\tau_n}\big| \cdot \big|\Delta X_{\tau_n}\big| \leq$$
$$\leq \big|(H_n)_{\tau_n}\big| \cdot \big|\Delta X_{\tau_n}\big| = \big|\Delta(H_n \bullet X)_{\tau_n}\big|,$$

$$\operatorname{sign}\big(\Delta(K_n^0 \bullet X)_{\tau_n}\big) = \operatorname{sign}\big(\Delta(H_n \bullet X)_{\tau_n}\big).$$

By treating $\Delta(H_n \bullet X)_{\tau_n} \geq 0$ and $\Delta(H_n \bullet X)_{\tau_n} \leq 0$ separately and considering (3.2), we obtain

$$\big(K_n^0 \bullet X\big)_{\tau_n} \geq -\frac{3}{n} \quad \text{a.s.}$$

Thus, for $K_n = K_n^0 \cdot I([\![0, \tau_n]\!])$ where $[\![0, \tau_n]\!]$ denotes the stochastic interval, we get

$$K_n \bullet X \geq -\frac{3}{n} \quad \text{a.s.}$$

From (3.3) and (3.4), it follows that

$$P\big((K_n \bullet X)_\infty > \delta_1 - \frac{1}{n}\big) > \delta_2 - \frac{1}{n}.$$

Since integrands K_n $(n \in I\!N)$ are bounded, they are componentwise X-integrable. Thus, the sequence K_n realizes $(\overline{A})_B$ on X that contradicts the above assumption. □

Theorem 3.5 The condition $(\widetilde{NA})_B$ implies $(\overline{NA})_V$.

Proof. Suppose that X satisfies $(\widetilde{NA})_B$ but does not satisfy $(NA)_V$, i.e., there exists an integrand H that realizes $(A)_V$ on X. Set

$$H_m = H \cdot I(|H| \leq m),$$
$$\tau_n = \inf\big\{t \geq 0 : (H_m \bullet X)_t \leq (H \bullet X) - a\big\}$$

where a is chosen from the inequality:

$$H \bullet X \geq -a \quad \text{a.s.}$$

As in the proof of the above proposition, it is easy to show that the sequence of integrands $K_m = H_m \cdot I([\![0, \tau_m]\!])$ realizes $(\widetilde{A})_B$ on X. Thus, $(\widetilde{NA})_B$ implies $(NA)_V$, and by applying Proposition 3.1, we complete the proof. $\square$

Corollary 3.6 There is equivalence between:

(*a*) X satisfies $(\widetilde{NA})_B$.

(*b*) X satisfies $(\widetilde{NA})_C$.

(*c*) There exists a probability measure $Q \sim P$ such that X is a martingale transform with respect to Q.

4 Example

This section presents an example of a semimartingale that satisfies $(\overline{NA})_C$ and has no equivalent measure such that X is a martingale transform with respect to this measure.

First, we construct two auxiliary processess A_t and B_t. Let $(\tau_i)_{i=1}^\infty$, $(\zeta_i)_{i=1}^\infty$, $(\rho_i)_{i=1}^\infty$ and $(\eta_i)_{i=1}^\infty$ be independent random variables such that τ_i, ρ_i are uniformly distributed on $[0,1]$, ζ_i are Gaussian with parameters $0,1$ and η_i take the values ± 1 with probability $1/2$. Set

$$
\begin{aligned}
T_n &= \sum_{i=1}^n \tau_i, \quad n \in I\!N, \\
S_n &= \sum_{i=1}^n \rho_i, \quad n \in I\!N, \\
A_t &= \sum_{\{n: T_n \le t\}} \zeta_n, \quad t \ge 0.
\end{aligned}
$$

Before constructing B_t, we introduce the process B_t^0 such that $B_0^0 = 0$,

$$
B^0_{S_n} = \begin{cases} 1 & \text{if } B^0_{S_{n-1}} = 1 \text{ or } \eta_n = 1, \\ -2 + \frac{1}{2^n} & \text{if } B^0_{S_{n-1}} \neq 1 \text{ and } \eta_n = -1 \end{cases}
$$

and B_t^0 is constant on intervals $[S_i, S_{i+1})$. Set

$$
\begin{aligned}
\varphi_t &= 1 - \{t\}, \\
B &= \varphi \bullet B^0
\end{aligned}
$$

where $\{t\}$ denotes the fractional part of t.

Let W_t be a standard linear Brownian motion being independent of the pair

$(A_t, B_t)_{t\geq 0}$. Set

$$\tilde{\mathcal{F}}^A_t = \sigma(A_s; s \leq t), \quad \mathcal{F}^A_t = \bigcap_{s>t} \tilde{\mathcal{F}}^A_s,$$
$$\tilde{\mathcal{F}}^B_t = \sigma(B_s; s \leq t), \quad \mathcal{F}^B_t = \bigcap_{s>t} \tilde{\mathcal{F}}^B_s,$$
$$\tilde{\mathcal{F}}^W_t = \sigma(W_s; s \leq t), \quad \mathcal{F}^W_t = \bigcap_{s>t} \tilde{\mathcal{F}}^W_s,$$
$$\mathcal{F}_t = \mathcal{F}^A_t \vee \mathcal{F}^B_t \vee \mathcal{F}^W_t, \quad \mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t.$$

It can be proved that $\mathcal{F}^A_t = \tilde{\mathcal{F}}^A_t$ and $\mathcal{F}^B_t = \tilde{\mathcal{F}}^B_t$ (see [7]).

The two-dimensional process X is given by

$$X^1_t = A_t + B_t + W_t,$$
$$X^2_t = -A_t + B_t - W_t.$$

This process is $\mathcal{F}_t$-semimartingale since A_t, B_t have finite variation and W_t is $\mathcal{F}_t$-Brownian motion due to the independence of $\mathcal{F}^A_t \vee \mathcal{F}^B_t$ and $\mathcal{F}^W_t$.

Proposition 4.1 The process X does not satisfy $(\overline{NA})_V$. Consequently, X possesses no equivalent measure for which X is a martingale transform.

Proof. Set

$$H^1_t = H^2_t = \frac{1}{\varphi_t}.$$

We have

$$\int_0^t |K_s| \, dVar(A_s \pm B_s) < \infty \quad \text{a.s.} \quad \forall t > 0 \tag{4.2}$$

for any one-dimensional process K and, in particular, for H^1 and H^2. Set $W^1_t = W_t$, $W^2_t = -W_t$ and let an increasing process C_t and $\mathcal{F}_t$-adapted processes π^{ij}_t satisfy the condition:

$$[W^i, W^j]_t = \int_0^t \pi^{ij}_s dC_s \quad (i, j = 1, 2).$$

The processes π^{ij} may be chosen so that $\pi^{11} = -\pi^{12} = -\pi^{21} = \pi^{22}$. Since $H^1 = H^2$, we get

$$\int_0^t \Big(\sum_{i,j=1}^2 H^i_s \pi^{ij}_s H^j_s \Big) dC_s = 0.$$

Together with (4.2), this implies that $H \in L(X)$. We may write

$$(H \bullet X)_t = \big((H^1 + H^2) \bullet B \big)_t = 2(1/\varphi \bullet B)_t = 2B^0_t$$

due to the definition of a vector stochastic integral and Theorem 2.4. From the construction of B^0_t, it follows that

$$B^0_t \geq -2, \qquad \lim_{t\to\infty} B^0_t = 1 \quad \text{a.s.}$$

Thus, X does not satisfy $(NA)_V$. Consequently, X does not satisfy $(\overline{NA})_V$. $\square$

Now we turn to the proof that X satisfies $(\overline{NA})_C$.

Lemma 4.3 Suppose that $H = (H_t^1, H_t^2)$ satisfies the following conditions:

$$H^i \in L(X^i) \quad (i = 1, 2),$$
$$H \bullet X \geq -a \quad \text{a.s.} \quad \text{for some } a \geq 0.$$

Then
$$H \bullet X = (H^1 + H^2) \bullet B.$$

Proof. From (4.2), it follows that $K \in L(A)$, $K \in L(B)$ for any one-dimensional process K. By Theorem 2.3, we obtain that $H^i \in L(W)$ $(i = 1, 2)$. Hence

$$H \bullet X = K^1 \bullet A + K^2 \bullet B + K^1 \bullet W$$

where $K^1 = H^1 - H^2$, $K^2 = H^1 + H^2$.

Let $\bar{P}$ be an (infinite) measure on $\big(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)\big)$, defined as $P \otimes \mu_L(\mathbb{R}_+)$ where $\mu_L(\mathbb{R}_+)$ denotes the Lebesgue measure on $\mathbb{R}_+$. Suppose that $\bar{P}(K^1 \neq 0) > 0$. Then, there exists $n \in \mathbb{N}$ such that

$$\bar{P}\big(K^1 \cdot I([\![T_{n-1}, T_n]\!]) \neq 0\big) > 0.$$

According to our construction, we may write

$$(4.4) \qquad \big(\Omega, \mathcal{G}, P\big) = \big(\Omega_1 \times \Omega_2, \mathcal{G}_1 \otimes \mathcal{G}_2, P_1 \otimes P_2\big)$$

where

$$\mathcal{G} = \mathcal{F},$$
$$\mathcal{G}_1 = \sigma\big(T_1, \ldots, T_{n-1}, \zeta_1, \ldots, \zeta_{n-1}\big) \vee \sigma\big(B_t; t \geq 0\big) \vee \sigma\big(W_t; t \geq 0\big),$$
$$\mathcal{G}_2 = \sigma\big(\tau_n, \tau_{n+1}, \ldots; \zeta_n, \zeta_{n+1}, \ldots\big).$$

We shall first prove that for any $\mathcal{F}_t$-predictable process K, the stopped process $K_t^{T_n} = K_{t \wedge T_n}$ is $\mathcal{G}_1 \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. For a fixed $s \geq 0$, we introduce the set

$$D = \{\omega : T_n(\omega) > s\}.$$

From the equality $\mathcal{F}_t^A = \tilde{\mathcal{F}}_t^A$, it follows that

$$\mathcal{F}_s|_D = \tilde{\mathcal{F}}_s^A|_D \vee \mathcal{F}_s^B|_D \vee \mathcal{F}_s^W|_D \subset$$
$$\subset \sigma\big(T_1, \ldots, T_{n-1}, \zeta_1, \ldots, \zeta_{n-1}\big)|_D \vee \sigma\big(B_t; t \geq 0\big)|_D \vee \sigma\big(W_t; t \geq 0\big)|_D \subset \mathcal{G}_1|_D .$$

If $K = I\big(E \times (s, \infty)\big)$ where $E \in \mathcal{F}_s$, then

$$K^{T_n} = I\Big((E \cap \{T_n > s\}) \times (s, \infty)\Big).$$

Such processes are obviously $\mathcal{G}_1 \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. The sets $E \times (s, \infty)$ where $s \geq 0, E \in \mathcal{F}_s$, together with the sets $E \times \{0\}$ where $E \in \mathcal{F}_0$, generate the predictable

σ-field on $\Omega \times I\!R_+$ (see [8]). Using the monotone class theorem, we deduce that K^{T_n} is $\mathcal{G}_1 \otimes \mathcal{B}(I\!R_+)$-measurable for any K being an indicator of a predictable set. The assertion for any predictable K can be proved with the help of a standard approximation procedure.

We shall now prove that $P\big(K^1_{T_n} \neq 0\big) > 0$. Set

$$\big(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{P}\big) = \big(\Omega \times [0,1],\ \mathcal{G} \otimes \mathcal{B}([0,1]),\ P \otimes \mu_l([0,1])\big),$$
$$\tilde{K}^1(\tilde{\omega}) = \tilde{K}^1(\omega, s) = (K^1)^{T_n}(\omega, T_{n-1}(\omega) + s), \quad s \in [0,1].$$

Due to $\mathcal{G}_1 \otimes \mathcal{B}([0,1])$-measurability of $\tilde{K}^1$, we may write

$$\tilde{K}^1(\tilde{\omega}) = \tilde{K}^1(\omega_1, \omega_2, s) = \tilde{K}^1(\omega_1, s).$$

Here, we use (4.4): $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Therefore

$$K^1_{T_n}(\omega) = (K^1)^{T_n}_{T_n}(\omega) = (K^1)^{T_n}\big(\omega, T_{n-1}(\omega) + \tau_n(\omega)\big) = \\ = \tilde{K}^1(\omega, \tau_n(\omega)) = \tilde{K}^1(\omega_1, \tau_n(\omega_2))$$

since τ_n is $\mathcal{G}_2$-measurable. Let θ be the mapping defined as

$$\theta : \Omega_1 \times \Omega_2 \ni (\omega_1, \omega_2) \longmapsto (\omega_1, \tau_n(\omega_2)) \in \Omega_1 \times [0,1].$$

It is easy to verify that $P_1 \otimes \mu_L = (P_1 \otimes P_2) \circ \theta^{-1}$ where $\mu_L = \mu_L([0,1])$. Thus, for any $D \in \mathcal{G}_1 \otimes \mathcal{B}([0,1])$, we have

$$P\big(\{\omega : (\omega, \tau_n(\omega)) \in D\}\big) = \\ = (P_1 \otimes P_2)\ (\{(\omega_1, \omega_2) : (\omega_1, \tau_n(\omega_2)) \in D\}) = \\ = (P_1 \otimes \mu_L)\ (D) = \tilde{P}(D).$$

Consequently

$$P\big(K^1_{T_n} \neq 0\big) = P\big(\{\omega : (\omega, \tau_n(\omega)) \in \{\tilde{K}^1 \neq 0\}\}\big) = \\ = \tilde{P}\big(\tilde{K}^1 \neq 0\big) \geq \bar{P}\big(K^1 \cdot I([\![T_{n-1}, T_n]\!]) \neq 0\big) > 0.$$

The random variable

$$(H \bullet X)_{T_n-} = \lim_{t \uparrow T_n} (H \bullet X)_t$$

is $\mathcal{F}_{T_n-}$-measurable (see [8]). As mentioned above, for any $s \geq 0, E \in \mathcal{F}_s$, we have $E \cap \{T_n > s\} \in \mathcal{G}_1$. Therefore, $\mathcal{F}_{T_n-} \subset \mathcal{G}_1$. Since r.v. $(\tau_i)_{i=1}^\infty$, $(\rho_i)_{i=1}^\infty$ are independent and uniformly distributed on $[0,1]$, we get

$$P\big(\{\omega : T_n(\omega) = S_m(\omega)\}\big) = 0$$

for any $m \in I\!N$. Together with continuity of W_t, this implies that

$$(H \bullet X)_{T_n} = (H \bullet X)_{T_n-} + K^1_{T_n} \cdot \Delta A_{T_n} = (H \bullet X)_{T_n-} + K^1_{T_n} \cdot \zeta_n \quad \text{a.s..}$$

We may find constants $M > 0, \delta > 0$ such that

$$P\Big(\big\{|(H \bullet X)_{T_n-}| < M\big\} \cap \big\{|K^1_{T_n}| > \delta\big\}\Big) > 0.$$

Due to unboundedness of ζ_n and independence of ζ_n and $\mathcal{G}_1$, we get

$$P\big((H \bullet X)_{T_n} < -a\big) > 0$$

that contradicts the hypothesis of the lemma. Thus, $K^1 = 0$ $\bar{P}-$a.s. and consequently, $K^1 \bullet W = 0$. The proof of this lemma also shows that $P(K^1_{T_n} \neq 0) = 0$ for any $n \in I\!N$, which implies the equality $K^1 \bullet A = 0$. Eventually

$$H \bullet X = K^2 \bullet B = (H^1 + H^2) \bullet B. \quad \square$$

Theorem 4.5 The process X satisfies $(\overline{NA})_C$. Consequently, X satisfies $(\overline{NA})_B$.

Proof. Suppose that X does not satisfy $(\overline{NA})_C$. Let $H_l = (H^1_{lt}, H^2_{lt})$ be a sequence of integrands realizing $(\overline{A})_C$ on X. By Lemma 4.3 and Theorem 2.4

$$H_l \bullet X = K_l \bullet B = K^0_l \bullet B^0 \tag{4.6}$$

where $K_l = H^1_l + H^2_l$, $K^0_l = \varphi \cdot K_l$. There exist constants $\delta_1 > 0, \delta_2 > 0, n \in I\!N$ such that

$$P\Big((K^0_l \bullet B^0)_{S_n} > \delta_1\Big) > \delta_2 \quad \forall l \in I\!N. \tag{4.7}$$

We may write

$$\big(\Omega, \mathcal{G}, P\big) = \big(\Omega_1 \times \Omega_2, \mathcal{G}_1 \otimes \mathcal{G}_2, P_1 \otimes P_2\big)$$

where

$$\mathcal{G}_1 = \sigma\big(\rho_1, \rho_2, \ldots, ; \eta_{n+1}, \eta_{n+2}, \ldots\big) \vee \sigma\big(A_t; t \geq 0) \vee \sigma\big(W_t; t \geq 0),$$
$$\mathcal{G}_2 = \sigma\big(\eta_1, \ldots, \eta_n\big).$$

There exists a probability measure $Q_2 \sim P_2$ such that r.v. $\eta_1, \ldots, \eta_n$ are independent with respect to Q_2 and the stopped process $(B^0)^{S_n}$ is a martingale with respect to $Q = P_1 \otimes Q_2$. We have

$$(K^0_l \bullet B^0)^{S_n} = (K^0_l \bullet (B^0)^{S_n}),$$

i.e., this process is a (one-dimensional) martingale transform. Since this process is bounded below, it is a Q-local martingale (see [1]) and by the Fatou lemma, is a Q-supermartingale. Together with (4.7), the equivalence between Q and P implies the existence of $l \in I\!N$ for which

$$Q\Big((H_l \bullet X)_{S_n} < -\frac{1}{l}\Big) > 0$$

Consequently, for some $\Delta > 0$, we have $P(D) > 0$ where

$$D = \Big\{(H_l \bullet X)_{S_n} < -\frac{1}{l} - \Delta\Big\}. \tag{4.8}$$

We may write

$$\big(\Omega, \mathcal{G}, P\big) = \big(\Omega_1 \times \Omega_2, \mathcal{G}_1 \otimes \mathcal{G}_2, P_1 \otimes P_2\big)$$

where

$$\mathcal{G}_1 = \sigma\big(S_1, \ldots, S_n, \eta_1, \ldots, \eta_n\big) \vee \sigma\big(A_t; t \geq 0\big) \vee \sigma\big(W_t; t \geq 0\big),$$
$$\mathcal{G}_2 = \sigma\big(\rho_{n+1}, \rho_{n+2}, \ldots; \eta_{n+1}, \eta_{n+2}, \ldots\big).$$

Set

$$\big(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{P}\big) = \big(\Omega \times [0,1],\ \mathcal{G} \otimes \mathcal{B}([0,1]),\ P \otimes \mu_l([0,1])\big),$$
$$\tilde{K}_l^0(\tilde{\omega}) = \tilde{K}_l^0(\omega, s) = (K_l^0)^{S_{n+1}}(\omega, S_n(\omega) + s), \quad s \in [0,1].$$

Suppose that

$$\tilde{P}\Big(\{\tilde{K}_l^0 < \Delta/2\} \cap (D \times [0,1])\Big) > 0$$

where D is defined in (4.8). Using reasoning similar to the proof of Lemma 4.3, one can show that the process $(K_l^0)^{S_{n+1}}$ and consequently, $(K_l^0)^{S_n}$ are $\mathcal{G}_1 \otimes \mathcal{B}(I\!R_+)$-measurable. Therefore, $D \in \mathcal{G}_1$ due to (4.6), (4.8) and $\mathcal{G}_1 \otimes \mathcal{B}(I\!R_+)$-measurability of B^0. As in the previous proof, we deduce that

$$P\Big(\{(K_l^0)_{S_{n+1}} < \Delta/2\} \cap D\Big) > 0.$$

Since η_{n+1} and $\mathcal{G}_1$ are independent, we get

$$P\Big(\{(K_l^0)_{S_{n+1}} < \Delta/2\} \cap D \cap \{\eta_{n+1} = +1\}\Big) > 0$$

which yields

$$P\Big(\Big\{(H_l \bullet X)_{S_{n+1}} < -\frac{1}{l}\Big\} \cap \Big\{\eta_{n+1} = +1\Big\}\Big) > 0.$$

As B is constant after S_{n+1} on the set $\{\eta_{n+1} = +1\}$, this inequality contradicts the condition

$$H_l \bullet X \geq -\frac{1}{l} \quad \text{a.s.}$$

Thus

$$K_l^0 \geq \Delta/2 \quad \text{on } [\![S_n, S_{n+1}]\!] \cap (D \times I\!R_+) \quad \bar{P} - \text{a.s.}$$

where $\bar{P} = P \otimes \mu_L(I\!R_+)$. We may find $m \in I\!N$ for which

$$P(S_n < m < S_{n+1}) > 0.$$

Consequently, there exists $\delta > 0$ such that

$$P\Big((K_l)_s \geq \frac{\Delta}{2(m-s)} \ \text{ for } \mu_l\text{-almost all } s \in (m-\delta, m)\Big) > 0. \tag{4.9}$$

Applying Theorem 2.3, we obtain that $K_l \in L(W)$. Thus, there exists a decomposition $W = A' + M'$ into a sum of a finite-variation process and a local martingale such that

$$\Big(\int_0^t (K_l)_s^2 d[M', M']_s\Big)^{1/2} \in \mathcal{A}_{loc}.$$

It is easy to verify the equality:

$$[M', M']_t = t + \sum_{s \le t} (\Delta A'_s)^2.$$

Therefore

$$\int_0^t (K_l)_s^2 ds < \infty \quad \text{a.s.} \quad \forall t \ge 0$$

that contradicts (4.9). This completes the proof. $\square$

Remark. Let X be an arbitrary d-dimensional semimartingale. It was already mentioned that any of the conditions $(\widetilde{NA})_V$, $(\widetilde{NA})_C$, $(\widetilde{NA})_B$, $(\overline{NA})_V$ implies the existence of an equivalent measure such that X is a martingale transform with respect to this measure. The condition $(NA)_V$ is not sufficient for the existence of such a measure even if $d = 1$ (see [2]). Together with our example, this shows that none of the conditions $(\overline{NA})_C$, $(\overline{NA})_B$, $(NA)_V$, $(NA)_C$, $(NA)_B$ implies the existence of an equivalent measure for which X is a martingale transform.

Acknowledgement. I would like to thank A.N. Shiryaev for setting up the problem and for many valuable remarks as well as helpful discussions.

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