WEIGHTED V@R AND ITS PROPERTIES

A.S. Cherny

Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Probability Theory,
119992 Moscow, Russia.
E-mail: `cherny@mech.math.msu.su`
Webpage: `http://mech.math.msu.su/~cherny`

Abstract. The paper deals with the study of the coherent risk measure, which we call *Weighted V@R*. It is a risk measure of the form

$$\rho_\mu(X) = \int_{[0,1]} \mathrm{TV@R}_\lambda(X)\mu(d\lambda),$$

where μ is a probability measure on $[0,1]$ and TV@R stands for Tail V@R.

After investigating some basic properties of this risk measure, we apply the obtained results to the financial problems of pricing, optimization, and capital allocation. It turns out that, under some regularity conditions on μ, Weighted V@R possesses some nice properties that are not shared by Tail V@R. To put it briefly, Weighted V@R is "smoother" than Tail V@R. This allows one to say that Weighted V@R is one of the most important classes (or maybe the most important class) of coherent risk measures.

Key words and phrases. Capital allocation, coherent risk measures, determining set, distorted measures, minimal extreme measure, No Good Deals pricing, spectral risk measures, strict diversification, Tail V@R, Weighted V@R.

Contents

1 Introduction

Historic overview. The theory of coherent risk measures is a very new, important, and rapidly evolving branch of the modern financial mathematics. This concept was introduced by Artzner, Delbaen, Eber, and Heath [4], [5]. Since then, many papers on the topic have followed; surveys of the modern state of the theory are given in [21], [25; Ch. 4], and [36]. In some sources theory of coherent risk measures and related topics is already called the "third revolution in finance" (see [39]).

A very important class of coherent risk measures is given by *Tail V@R* (the terms *Average V@R*, *Conditional V@R*, and *Expected Shortfall* are also used). Tail V@R of order $\lambda \in [0,1]$ is a map $\rho_\lambda : L^\infty \to \mathbb{R}$ (we have a fixed probability space $(\Omega, \mathcal{F}, \mathsf{P})$) defined by

$$\rho_\lambda(X) = -\inf_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X,$$

where $\mathcal{D}_\lambda$ is the set of probability measures Q that are absolutely continuous with respect to P with $\frac{d\mathsf{Q}}{d\mathsf{P}} \le \lambda^{-1}$. (From the financial point of view, X is the discounted cash flow of some financial transaction.) The importance of Tail V@R is seen from a result of Kusuoka [28], who proved that ρ_λ is the smallest law invariant coherent risk measure that dominates V@R$_\lambda$ (we recall the precise formulation in Section 2). This suggests an opinion that Tail V@R is one of the best coherent risk measures. For more information on Tail V@R, see [3], [20; Sect. 6], [21; Sect. 7], [25; Sect. 4.4], [36; Sect. 1.3].

However, there exists a risk measure, which is, in our opinion, much better than Tail V@R. It is given by

$$\rho_\mu(X) = \int_{[0,1]} \rho_\lambda(X)\mu(d\lambda), \tag{1.1}$$

where μ is a probability measure on $[0,1]$. We call this risk measure *Weighted V@R* and its study is the goal of this paper. First of all, let us give two arguments in favor of Weighted V@R over Tail V@R:

- (Financial argument) Tail V@R of order λ takes into consideration only the λ-tail of the distribution of X; thus, two distributions with the same λ-tail will be assessed by this measure in the same way, although one of them might be clearly better than the other (see Figure 1). On the other hand, if the right endpoint of $\operatorname{supp}\mu$ is 1, then ρ_μ depends on the whole distribution of X.
- (Mathematical argument) If the weighting measure μ satisfies the condition $\operatorname{supp}\mu = [0,1]$, then ρ_μ possesses some nice properties that are not shared by ρ_λ. In particular, various optimization problems have a unique solution (see Section 5).

The paper [23] provides some further financial arguments in favor of Weighted V@R.

It might seem rather surprising that the risk measure, which we call here Weighted V@R, was considered by actuaries already in the early 90s, i.e. before the papers of Artzner, Delbaen, Eber, and Heath; see, for example, [22], [40] (see also the paper [41], which appeared at the same time as [4]). These papers are related to the object termed *distorted measure*. This is a functional on random variables defined as

$$\rho(X) = \int_{-\infty}^{0} \Psi(F(x))dx + \int_{0}^{\infty} (\Psi(F(x)) - 1)dx, \tag{1.2}$$

where $\Psi : [0,1] \to [0,1]$ is an increasing concave function with the properties $\Psi(0) = 0$, $\Psi(1) = 1$ and F is the distribution function of X. It turns out that the class of these functionals (with different Ψ) is exactly the class of Weighted V@Rs (with different μ).

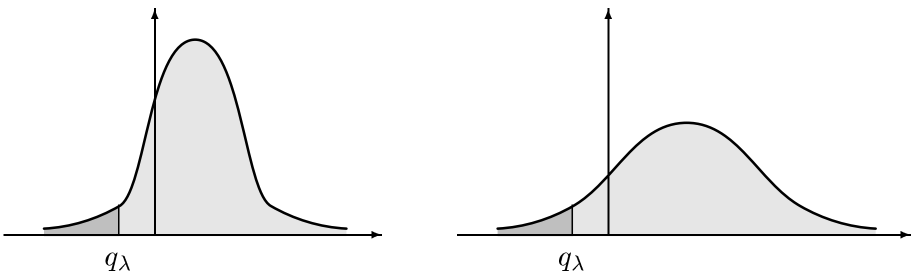

Figure 1. These two distributions have the same λ-tails (q_λ is the λ-quantile), so that $\mathrm{TV@R}_\lambda$ coincides on them. However, the distribution at the right stochastically dominates the distribution at the left.

For risk measures on L^∞, this equivalence can be found in [25; Th. 4.64] or [36; Th. 1.51]; for risk measures on L^0, this equivalence is proved in Section 3 of this paper.

The first appearance of ρ_μ in the framework of coherent risk measures is in the paper of Kusuoka [28]. He proved that any law invariant comonotonic coherent risk measure is of this form. In the same paper, he proved that any law invariant risk measure has the form $\sup_{\mu\in\mathfrak{M}}\rho_\mu$, where $\mathfrak{M}$ is a set of probability measures on $[0,1]$ (we recall the precise formulations in Section 2).

Some further considerations of ρ_μ can be found in the papers of Acerbi [1], [2], who uses the term *spectral risk measures* for this class.

Furthermore, Carlier and Dana [11] provided a representation of the determining set of Weighted V@R (the definition of this notion is given below). This representation is recalled in Section 4.

Structure of the paper. According to the classical definition, a coherent risk measure is defined on bounded random variables. However, for financial applications it is almost necessary to extend this notion to the space of all random variables. Indeed, most distributions used in theory (for example, the lognormal one) are unbounded. In this paper, we consider coherent risk measures defined on the space L^0 of all random variables. This is done as follows. According to the basic representation theorem, a functional $\rho : L^\infty \to \mathbb{R}$ is a coherent risk measure if and only if it admits a representation

$$\rho(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q}X, \quad X \in L^\infty \tag{1.3}$$

with some set $\mathcal{D}$ of probability measures that are absolutely continuous with respect to P. For finite Ω, this was proved in [5]; for arbitrary Ω, this was proved by Delbaen [20]; in the latter case, an additional continuity assumption called the Fatou property should be imposed on ρ. Following [16], [17], we take representation (1.3) with L^∞ replaced by L^0 as the definition of a coherent risk measure on L^0. The expectation $\mathsf{E}_\mathsf{Q}X$ is understood as $\mathsf{E}_\mathsf{Q}X^+ - \mathsf{E}_\mathsf{Q}X^-$ with the convention $\infty - \infty = -\infty$ so that $\mathsf{E}_\mathsf{Q}X$ is well defined for any Q and X. Thus, ρ takes on values in the extended real line $[-\infty,\infty]$.

Section 2 contains some basic definitions as well as known results on Tail V@R and Weighted V@R.

In Section 3, we provide two representations of Weighted V@R. These are the extensions of (1.1) and (1.2) to L^0.

Clearly, different sets $\mathcal{D}$ might define the same coherent risk measure. However, among all the sets that define the same risk measure ρ there exists the largest one (it has the form $\mathcal{D} = \{\mathsf{Q} \ll \mathsf{P} : \mathsf{E}_\mathsf{Q}X \geq -\rho(X) \text{ for any } X\}$). We call it the *determining set* of ρ.

Finding the structure of this set is important. For example, in [16; Sect. 2], we considered the No Good Deals pricing technique based on coherent risk measures. According to this technique, the interval of fair prices of a contingent claim F is $\{\mathsf{E}_\mathsf{Q}F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}$, where $\mathcal{R}$ is the set of risk-neutral measures and $\mathcal{D}$ is the determining set of a risk measure ρ, which serves as an input to this technique. Furthermore, the solutions of various other financial problems given in [16], [17] are expressed through the determining set. In Section 4, we provide two representations of the determining set of Weighted V@R, both of which are given in different terms than the one in [11].

The main result of Section 5 is that

$$\rho_\mu(X+Y) < \rho_\mu(X) + \rho_\mu(Y)$$

provided that $\operatorname{supp} \mu = [0,1]$ and X, Y are not comonotone (in particular, the latter condition is satisfied if the distribution of (X,Y) has a joint density). We call this the *strict diversification property.* This property is very important from the viewpoint of financial mathematics because it leads to the uniqueness of a solution of various optimization problems based on coherent risk measures.

In [16], we introduced the notion of an *extreme measure.* The class of extreme measures for a coherent risk measure ρ and a random variable X is defined as

$$\mathcal{X}_\rho(X) = \Big\{\mathsf{Q} \in \mathcal{D} : \mathsf{E}_\mathsf{Q}X = \inf_{\mathsf{Q}' \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}'}X \in (-\infty, \infty)\Big\},$$

where $\mathcal{D}$ is the determining set of ρ. This notion was found to be very convenient and important. In particular, the solutions of several optimality pricing problems, the solution of the equilibrium pricing problem, and the solution of the capital allocation problem are expressed through extreme measures (see [16], [17]). Moreover, the *risk contribution* introduced in [16] is expressed through extreme measures. In general, the set $\mathcal{X}_\rho(X)$ can contain more than one point. However, as shown in Section 6, for $\rho = \rho_\mu$ with $\mu(\{0\}) = 0$, there exists a unique element of $\mathcal{X}_\rho$ that is the smallest in the convex order. We call it the *minimal extreme measure.* This notion is of importance for financial mathematics as it allows one to select a (unique) distinguished solution of the problems like capital allocation or optimality pricing, which possesses some nice properties. We call it the *central solution.*

One of the most important goals of the modern financial mathematics is to narrow the No Arbitrage price intervals of contingent claims as they are known to be unacceptably wide in most incomplete models (see, for example, the discussion in [19; Sect. 5]). Several ways to do that have been proposed in the literature. One of them consists in considering actively traded derivatives as basic assets. In particular, a popular model is based on treating as basic assets the European call options on a fixed asset with a fixed maturity and different strike prices. The corresponding model was first studied by Breeden and Litzenberger [9] and Banz and Miller [7]. A literature review of this model is given in [26]. Let us also mention the paper [14; Sect. 6], in which this model was analyzed from the general viewpoint of fundamental theorems of asset pricing.

Recently, another (very promising) way to narrow fair price intervals has been proposed. It is known as the *No Good Deals* pricing. This technique was first considered by Cochrane and Saá-Requejo [18] and Bernardo and Ledoit [8]. An important feature of this theory is that there exists no canonical definition of a good deal (in particular, [18] and [8] employ different definitions). Carr, Geman, and Madan [12] (see also the review paper [13]) and Jaschke and Küchler [27] proposed variants of No Good Deals pricing based on coherent risk measures. These techniques were further developed in [16] and [38].

In Section 7, we combine the two ways of narrowing fair price intervals described above. Namely, we apply the No Good Deals pricing technique from [16] to the model with European options as basic assets. This leads to the "double reduction" of fair price intervals. The risk measure employed is Weighted V@R. In fact, the results of Section 7 provide a description of risk-neutral densities that price correctly traded call options and are "not far" from the real-world density, the "distance" being measured with the help of a coherent risk measure. Let us remark that the study of the interplay between risk-neutral and real-world densities is a very popular topic of the modern financial mathematics (see, in particular, the papers [6], [10], [29]).

Section 8 deals with the Markowitz-type optimization problem of the form

$$\begin{cases} \mathsf{E}_{\mathsf{P}} X \longrightarrow \max, \\ X \in A, \ \rho(X) \le c, \end{cases} \tag{1.4}$$

where ρ is a coherent risk measure, $c \ge 0$, and A is the set of possible discounted cash flows that can be obtained by using various trading strategies in the model under consideration. Let us remark that in [31] Markowitz proposed to consider an alternative of his mean-variance optimization problem with variance replaced by semivariance $S(X) = \mathsf{E}((X - \mathsf{E}X)^-)^2$ (clearly, variance is not a good measure of risk because it punishes profits in the same way as losses). In (1.4), the risk is measured in a coherent way. Let us remark that problems of type (1.4) were considered in [2], [17], [32], [33].

Here we provide a solution of (1.4) for the case, where ρ is Weighted V@R and the model is complete, i.e. $A = \{X \in L^1(\mathsf{Q}) : \mathsf{E}_{\mathsf{Q}} X = 0\}$, where Q is a given measure (the unique risk-neutral measure). Classical examples of a complete model are the Black-Scholes model and the Cox-Ross-Rubinstein model. One more example is the option-based model considered in Section 7 (if we assume that the options on a basic asset with a fixed maturity and all the positive strikes are traded, then this model is complete). Our solution shows that the optimal strategy typically consists in buying binary options with the payoff $I\big(\frac{d\mathsf{Q}}{d\mathsf{P}} \le c_*\big)$, where c_* is the optimal threshold explicitly calculated in Section 8 (see Figure 5).

2 Basic Definitions

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. It will be convenient for us to deal not with coherent risk measures, but with their opposites called *coherent utility functions* (this enables one to get rid of numerous minus signs).

Definition 2.1. A *coherent utility function on* L^∞ is a map $u : L^\infty \to \mathbb{R}$ with the properties:

(a) (Superadditivity) $u(X+Y) \ge u(X) + u(Y)$;
(b) (Monotonicity) If $X \le Y$, then $u(X) \le u(Y)$;
(c) (Positive homogeneity) $u(\lambda X) = \lambda u(X)$ for $\lambda \in \mathbb{R}_+$;
(d) (Translation invariance) $u(X+m) = u(X) + m$ for $m \in \mathbb{R}$;
(e) (Fatou property) If $|X_n| \le 1$, $X_n \xrightarrow{\mathsf{P}} X$, then $u(X) \ge \limsup_n u(X_n)$.

The corresponding *coherent risk measure* is $\rho(X) = -u(X)$.

The theorem below was established in [5] for the case of a finite Ω (in this case the axiom (e) is not needed) and in [20] for the general case. We denote by $\mathcal{P}$ the set of probability measures on $\mathcal{F}$ that are absolutely continuous with respect to P. Throughout the paper, we identify measures from $\mathcal{P}$ (these are typically denoted by Q) with their densities with respect to P (these are typically denoted by Z).

Theorem 2.2. *A function u satisfies conditions* (a)–(e) *if and only if there exists a nonempty set $\mathcal{D} \subseteq \mathcal{P}$ such that*

$$u(X) = \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X, \quad X \in L^\infty. \tag{2.1}$$

Now, we use representation (2.1) to extend coherent utility functions to the space L^0 of all random variables.

Definition 2.3. A *coherent utility function on L^0* is a map $u : L^0 \to [-\infty, \infty]$ defined as

$$u(X) = \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0, \tag{2.2}$$

where $\mathcal{D}$ is a nonempty subset of $\mathcal{P}$ and $\mathsf{E}_\mathsf{Q} X$ is understood as $\mathsf{E}_\mathsf{Q} X^+ - \mathsf{E}_\mathsf{Q} X^-$ with the convention $\infty - \infty = -\infty$.

Clearly, a set $\mathcal{D}$, for which representations (2.1) and (2.2) are true, is not unique. However, there exists the largest such set given by $\{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X \geq u(X) \text{ for any } X\}$.

Definition 2.4. We will call the largest set, for which (2.1) (resp., (2.2)) is true, the *determining set of u*.

`Remarks`. (i) Clearly, the determining set is convex. For coherent utility functions on L^∞, it is also L^1-closed. However, for coherent utility functions on L^0, it is not necessarily L^1-closed. As an example, take a positive unbounded random variable X_0 such that $\mathsf{P}(X_0 = 0) > 0$ and consider $\mathcal{D}_0 = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X_0 = 1\}$. Clearly, the determining set $\mathcal{D}$ of the coherent utility function $u(X) = \inf_{\mathsf{Q}\in\mathcal{D}_0} \mathsf{E}_\mathsf{Q} X$ satisfies $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X_0 \geq 1\}$. On the other hand, the L^1-closure of $\mathcal{D}_0$ contains a measure Q_0 concentrated on $\{X_0 = 0\}$.

(ii) Let $\mathcal{D}$ be an L^1-closed convex subset of $\mathcal{P}$. Define a coherent utility function u by (2.1) or (2.2). Then $\mathcal{D}$ is the determining set of u. Indeed, assume that the determining set $\widetilde{\mathcal{D}}$ is greater than $\mathcal{D}$, i.e. there exists $\mathsf{Q}_0 \in \widetilde{\mathcal{D}} \setminus \mathcal{D}$. Then, by the Hahn-Banach theorem, we can find $X_0 \in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0} X_0 < \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$, which is a contradiction.

Now, we recall some basic facts related to Tail V@R. The next definition applies both to L^∞ and to L^0.

Definition 2.5. *Tail V@R* is the risk measure corresponding to the coherent utility function

$$u_\lambda(X) = \inf_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X,$$

where $\lambda \in [0, 1]$ and

$$\mathcal{D}_\lambda = \left\{ \mathsf{Q} \in \mathcal{P} : \frac{d\mathsf{Q}}{d\mathsf{P}} \leq \lambda^{-1} \right\}.$$

Clearly, $u_0(X) = \operatorname{essinf}_\omega X(\omega)$. The following well-known proposition provides two representations of Tail V@R with $\lambda > 0$. Throughout the paper, we denote by $q_\lambda(X)$ the right λ-quantile of X, i.e. $q_\lambda(X) = \inf\{x : \mathsf{P}(X \leq x) > \lambda\}$ (we use the convention $\inf \emptyset = +\infty$).

Proposition 2.6. **(i)** *Let* $\lambda \in (0,1]$, $X \in L^0$. *Then, for any* $Z^* \in \mathcal{D}_\lambda$ *such that*

$$Z^* = \begin{cases} \lambda^{-1} & \text{on } \{X < q_\lambda(X)\}, \\ 0 & \text{on } \{X > q_\lambda(X)\}, \end{cases} \tag{2.3}$$

we have $u_\lambda(X) = \mathsf{E}_\mathsf{P} XZ^*$. *Conversely, if* $u_\lambda(X) > -\infty$, *then any* $Z^* \in \mathcal{D}_\lambda$ *such that* $u_\lambda(X) = \mathsf{E}_\mathsf{P} XZ^*$, *should satisfy* (2.3).

(ii) *Let* $\lambda \in (0,1]$, $X \in L^0$. *Then*

$$u_\lambda(X) = \lambda^{-1} \int_{(-\infty, q_\lambda(X))} x \mathsf{Q}(dx) + c q_\lambda(X),$$

where $\mathsf{Q} = \mathrm{Law}_\mathsf{P} X$ *and* $c = 1 - \lambda^{-1}\mathsf{Q}((-\infty, q_\lambda(X)))$.

`Proof`. **(i)** We will assume that $\mathsf{E}_\mathsf{P} X^- < \infty$ and $\lambda \in (0,1)$ (the other cases are analyzed trivially). Without loss of generality, $q_\lambda(X) = 0$. Then, for any $Z \in \mathcal{D}_\lambda$,

$$XZ - XZ^* = X(Z - \lambda^{-1}) I(X < 0) + XZI(X > 0) \geq 0.$$

Furthermore, the a.s. inequality here is possible only if Z satisfies (2.3).

(ii) This is an immediate consequence of (i). □

The importance of Tail V@R is seen from the result of Kusuoka [28], which is stated below (its proof can also be found in [25; Th. 4.61] or [36; Th. 1.48]). Recall that a coherent utility function u is called *law invariant* if $u(X)$ depends only on the distribution of X.

Theorem 2.7. *Assume that* $(\Omega, \mathcal{F}, \mathsf{P})$ *is atomless. Let* $\lambda \in [0,1]$ *and* u *be a law invariant coherent utility function on* L^∞ *such that* $u \leq q_\lambda$. *Then* $u \leq u_\lambda$.

We now introduce the basic object of the paper.

Definition 2.8. **(i)** *Weighted V@R on* L^∞ is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \int_{[0,1]} u_\lambda(X) \mu(d\lambda), \quad X \in L^\infty,$$

where μ is a probability measure on $[0,1]$.

(ii) *Weighted V@R on* L^0 is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \inf_{\mathsf{Q} \in \mathcal{D}_\mu} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0,$$

where $\mathcal{D}_\mu$ is the determining set of u_μ on L^∞.

Thus, we have used the following scheme to define u_μ on L^0:

$$u_\lambda \text{ on } L^\infty \longrightarrow u_\mu \text{ on } L^\infty \longrightarrow \mathcal{D}_\mu \longrightarrow u_\mu \text{ on } L^0.$$

Let us now recall two results of Kusuoka [28] (the proofs can also be found in [25; Cor. 4.58, Th. 4.87] or [36; Cor. 1.45, Th. 1.58]), which show the importance of Weighted V@R in view of the law invariance property. Recall that random variables X and Y are *comonotone* if $(X(\omega_2) - X(\omega_1))(Y(\omega_2) - Y(\omega_1)) \geq 0$ for $\mathsf{P} \times \mathsf{P}$-a.e. ω_1, ω_2; a coherent utility function u is *comonotonic* if $u(X+Y) = u(X) + u(Y)$ whenever X and Y are comonotone.

Theorem 2.9. **(i)** *On an atomless probability space, a coherent utility function u on L^∞ is law invariant and comonotonic if and only if it has the form $u = u_\mu$ with some probability measure μ.*

(ii) *On an atomless probability space, a coherent utility function u on L^∞ is law invariant if and only if it has the form $u = \inf_{\mu \in \mathfrak{M}} u_\mu$ with some collection $\mathfrak{M}$ of probability measures on $[0,1]$.*

`Remark`. It is easy to check that Tail V@R is in fact a weighted average of V@Rs: $u_\lambda(X) = \int_0^\lambda q_s(X)ds$. Hence, Weighted V@R is also a weighted average of V@Rs, which supports the term we are using (otherwise, we should have called it "Weighted Tail V@R").

3 Representation of Weighted V@R

For $X \in L^0$ and $\lambda \in (0,1]$, we set

$$Z_\lambda^*(X) = \begin{cases} \lambda^{-1} & \text{on } \{X < q_\lambda(X)\}, \\ c & \text{on } \{X = q_\lambda(X)\}, \\ 0 & \text{on } \{X > q_\lambda(X)\}, \end{cases} \tag{3.1}$$

where $c \in [0, \lambda^{-1}]$ is such that $\mathsf{E_P} Z_\lambda^* = 1$.

Lemma 3.1. *Let $\lambda \in (0,1]$, $X \in L^0$, and f be an increasing function. Then $u_\lambda(f(X)) = \mathsf{E_P} f(X) Z_\lambda^*(X)$.*

`Proof`. Without loss of generality, $q_\lambda(X) = 0$ and $f(0) = 0$. Then, for any $Z \in \mathcal{D}_\lambda$, we can write

$$f(X)Z - f(X)Z_\lambda^*(X) = f(X)(Z - \lambda^{-1})I(X < 0) + f(X)ZI(X > 0) \geq 0.$$

Theorem 3.2. **(i)** *Suppose that $\mu(\{0\}) = 0$. Then, for $X \in L^0$, we have $u_\mu(X) = \mathsf{E_P} X Z_\mu^*(X)$, where $Z_\mu^*(X) = \int_{(0,1]} Z_\lambda^*(X)\mu(d\lambda)$.*

(ii) *We have*

$$u_\mu(X) = \int_{[0,1]} u_\lambda(X)\mu(d\lambda), \quad X \in L^0, \tag{3.2}$$

where $\int_{[0,1]} f(\lambda)\mu(d\lambda)$ is understood as $\int_{[0,1]} f^+(\lambda)\mu(d\lambda) - \int_{[0,1]} f^-(\lambda)\mu(d\lambda)$ with the convention $\infty - \infty = -\infty$.

`Proof`. **(i)** Any $Z \in \mathcal{D}_\mu$ can be represented as $\int_{(0,1]} Z_\lambda \mu(d\lambda)$ with $Z_\lambda \in \mathcal{D}_\lambda$ (see Theorem 4.4 below). Due to Lemma 3.1,

$$\begin{aligned} \mathsf{E_P}(m \vee X \wedge n)Z &= \int_{(0,1]} [\mathsf{E_P}(m \vee X \wedge n)Z_\lambda]\mu(d\lambda) \\ &\geq \int_{(0,1]} [\mathsf{E_P}(m \vee X \wedge n)Z_\lambda^*(X)]\mu(d\lambda) = \mathsf{E_P}(m \vee X \wedge n)Z_\mu^*(X), \quad m, n \in \mathbb{Z}. \end{aligned}$$

Obviously,

$$\mathsf{E_P} X Z_\mu^*(X) = \lim_{n \to \infty} \lim_{m \to -\infty} \mathsf{E_P}(m \vee X \wedge n) Z_\mu^*(X) \tag{3.3}$$

and the same is true for $Z_\mu^*(X)$ replaced by Z. Thus, $\mathsf{E_P} XZ \geq \mathsf{E_P} X Z_\mu^*(X)$, so that $u_\mu(X) = \mathsf{E_P} X Z_\mu^*(X)$.

(ii) Suppose first that $\mu(\{0\})=0$. Due to Lemma 3.1,

$$\mathsf{E}_\mathsf{P}(m\vee X\wedge n)Z_\mu^*(X)=\int_{(0,1]}u_\lambda(m\vee X\wedge n)\mu(d\lambda),\quad m,n\in\mathbb{Z}.$$

Obviously,

$$\int_{(0,1]}u_\lambda(X)\mu(d\lambda)=\lim_{n\to\infty}\lim_{m\to-\infty}\int_{(0,1]}u_\lambda(m\vee X\wedge n)\mu(d\lambda).$$

Combining this with (3.3) and the result of (i), we get (3.2).

Now, let $\mu(\{0\})=\alpha>0$. Then $\mu=\alpha\delta_0+(1-\alpha)\widetilde{\mu}$ and it follows from Theorem 4.4 that $\mathcal{D}_\mu=\alpha\mathcal{D}_{\delta_0}+(1-\alpha)\mathcal{D}_{\widetilde{\mu}}$. If X is not bounded below, then, clearly, both sides of (3.2) are equal to $-\infty$. If X is bounded below, then $u_\mu(X)=\alpha u_0(X)+(1-\alpha)u_{\widetilde{\mu}}(X)$ and equality (3.2) for μ follows from (3.2) for $\widetilde{\mu}$, which was proved above. □

In order to provide another representation of Weighted V@R, let us consider the function

$$\Psi_\mu(x)=\begin{cases}\mu(\{0\})+\displaystyle\int_0^x\int_{(y,1]}\lambda^{-1}\mu(d\lambda)dy, & x\in(0,1],\\ 0, & x=0.\end{cases}\tag{3.4}$$

Clearly, Ψ_μ is increasing, concave, $\Psi_\mu(0)=0$, and $\Psi_\mu(1)=1$. In fact, (3.4) establishes a one-to-one correspondence between functions with these properties and probability measures μ on $[0,1]$ (for details, see [25; Lem. 4.63] or [36; Lem. 1.50]). Further properties of Ψ_μ are:

$$\Psi_\mu(0+)=\mu(\{0\}),\qquad \Psi'_\mu(0+)=\int_{(0,1]}\lambda^{-1}\mu(d\lambda),\qquad \Psi'_\mu(1-)=\mu(\{1\}).$$

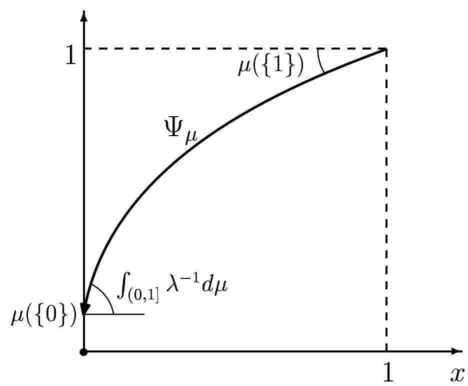

Figure 2. The structure of Ψ_μ

Theorem 3.3. *For $X\in L^0$,*

$$u_\mu(X)=-\int_{-\infty}^0\Psi_\mu(F(x))dx+\int_0^\infty(1-\Psi_\mu(F(x)))dx,\tag{3.5}$$

where F is the distribution function of X, and we use the convention $\infty-\infty=-\infty$.

`Proof`. It is seen from (3.2) that

$$u_\mu(X)=\lim_{n\to\infty}\lim_{m\to-\infty}u_\mu(X_{mn}),$$

where $X_{mn} = m \vee X \wedge n$. Similar limit relation holds for the right-hand side of (3.5) (one should consider the distribution function F_{mn} of X_{mn}). For bounded X, the statement of the theorem is known (see [25; Th. 4.64] or [36; Th. 1.51]), so that the result for general X is obtained by passing on to the limit. □

`Remarks`. (i) Some important regularity properties of μ can be expressed in terms of Ψ_μ. For example, $\Psi_\mu(0+) = 0$ if and only if $\mu(\{0\}) = 0$ (this condition will be important in Sections 6 and 7; note also that this condition is equivalent to the lower semi-continuity of u_μ on L^∞); Ψ_μ is strictly concave if and only if $\operatorname{supp}\mu = [0,1]$ (this condition will be important in Sections 5 and 8).

(ii) Integrating (3.5) by parts, we get

$$u_\mu(X) = \int_{\mathbb{R}} x d\Psi_\mu(F(x)) = \mathsf{E}_{\mathsf{P}} Y,$$

where Y is a random variable with the distribution function $\Psi_\mu \circ F$.

4 Representation of the Determining Set

We begin with three auxiliary lemmas. The notation $\mu \preccurlyeq \nu$ means that ν dominates μ in the monotone order, i.e. $\mu((-\infty, x]) \geq \nu((-\infty, x])$ for any x.

Lemma 4.1. *If $\mu \preccurlyeq \nu$, then $\mathcal{D}_\mu \supseteq \mathcal{D}_\nu$.*

`Proof`. There exist random variables ξ, η on some filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathsf{P}})$ such that $\operatorname{Law}\xi = \mu$, $\operatorname{Law}\eta = \nu$, and $\xi \leq \eta$ a.s. (see [37; § 1.A]). We can write $u_\mu(X) = \mathsf{E}_{\widetilde{\mathsf{P}}}\varphi(\xi)$, $u_\nu(X) = \mathsf{E}_{\widetilde{\mathsf{P}}}\varphi(\eta)$, where $\varphi(\lambda) = u_\lambda(X)$. As φ is increasing, $u_\mu \leq u_\nu$. Clearly, this implies that $\mathcal{D}_\mu \supseteq \mathcal{D}_\nu$. □

Lemma 4.2. *If μ_n tend to μ weakly and $\mu_n \preccurlyeq \mu$, then $\mathcal{D}_\mu = \bigcap_n \mathcal{D}_{\mu_n}$.*

`Proof`. Suppose that there exists $\mathsf{Q}_0 \in \bigcap_n \mathcal{D}_{\mu_n} \setminus \mathcal{D}_\mu$. As $\mathcal{D}_\mu$ is L^1-closed, we can apply the Hahn-Banach theorem, which yields $X_0 \in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0} X_0 < \inf_{\mathsf{Q} \in \mathcal{D}_\mu} \mathsf{E}_{\mathsf{Q}} X_0$. Thus, $\sup_n u_{\mu_n}(X_0) \leq \mathsf{E}_{\mathsf{Q}_0} X_0 < u_\mu(X_0)$. On the other hand, $u_{\mu_n}(X_0) \to u_\mu(X_0)$ since $u_\lambda(X_0)$ is continuous in λ. The obtained contradiction yields the inclusion $\mathcal{D}_\mu \supseteq \bigcap_n \mathcal{D}_{\mu_n}$. The reverse inclusion follows from the previous lemma. □

Lemma 4.3. *Let $\mu = \sum_{n=1}^N a_n \delta_{\lambda_n}$, where $\lambda_1 > \dots > \lambda_N \geq 0$. Then $\mathcal{D}_\mu = \sum_{n=1}^N a_n \mathcal{D}_{\lambda_n}$.*

`Proof`. With no loss of generality, $\lambda_N = 0$. Denote $\sum_{n=1}^N a_n \mathcal{D}_{\lambda_n}$ by $\mathcal{D}$. Clearly, $\mathcal{D}$ is convex. It is seen from Proposition 2.6 (i) that for any $X \in L^\infty$ the minimum of expectations $\mathsf{E}_{\mathsf{P}} XZ$ over $Z \in \widetilde{\mathcal{D}} := \sum_{n=1}^{N-1} a_n \mathcal{D}_n$ is attained. By the James theorem (see [24]), $\widetilde{\mathcal{D}}$ is weakly compact. We have $\mathcal{D} = \widetilde{\mathcal{D}} + a_N \mathcal{P}$, which is the sum of a convex weakly compact set and a convex weakly closed set. An application of the Hahn-Banach theorem shows that $\mathcal{D}$ is weakly closed. As $\mathcal{D}$ is convex, another application of the Hahn-Banach theorem shows that it is L^1-closed.

Obviously, $u_\mu(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}} X$ for any $X \in L^\infty$. Taking into account Remark (ii) following Definition 2.4, we get $\mathcal{D}_\mu = \mathcal{D}$. □

Theorem 4.4. *We have*

$$\mathcal{D}_\mu = \Big\{\int_{[0,1]} Z_\lambda \mu(d\lambda) : Z(\lambda,\omega) \text{ is jointly measurable} \\ \text{and } Z_\lambda \in \mathcal{D}_\lambda \text{ for any } \lambda \in [0,1]\Big\}. \tag{4.1}$$

`Proof`. Denote the right-hand side of (4.1) by $\mathcal{D}$. Set

$$\mu_n = \sum_{k=1}^{n-1} \mu\Big(\Big[\frac{k-1}{n}, \frac{k}{n}\Big)\Big)\delta_{\frac{k-1}{n}} + \mu\Big(\Big[\frac{n-1}{n}, 1\Big]\Big)\delta_1, \quad n \in \mathbb{N}.$$

Due to Lemma 4.3, $\mathcal{D} \subseteq \mathcal{D}_{\mu_n}$, and by Lemma 4.2, $\mathcal{D} \subseteq \mathcal{D}_\mu$.

Let us prove the reverse inclusion. Clearly, $\mathcal{D}$ is convex. Arguing in the same way as in the previous proof, we conclude that $\mathcal{D}$ is L^1-closed. Take

$$\mu_n = \mu\Big(\Big[0, \frac{1}{n}\Big]\Big)\delta_{\frac{1}{n}} + \sum_{k=2}^{n} \mu\Big(\Big(\frac{k-1}{n}, \frac{k}{n}\Big]\Big)\delta_{\frac{k}{n}}, \quad n \in \mathbb{N}.$$

Due to Lemma 4.3, $\mathcal{D}_{\mu_n} \subseteq \mathcal{D}$, and therefore, $u_{\mu_n} \ge u$, where $u(X) = \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$. As $u_{\mu_n}(X) \to u_\mu(X)$ for any $X \in L^\infty$, we get $u_\mu \ge u$ on L^∞. Employing the Hahn-Banach argument, we get $\mathcal{D}_\mu \subseteq \mathcal{D}$. □

Now, we describe another representation of $\mathcal{D}_\mu$, which was obtained by Carlier and Dana [11] (the proof can also be found in [25; Th. 4.73] or [36; Th. 1.53]).

Theorem 4.5. *We have*

$$\begin{aligned}\mathcal{D}_\mu &= \{\mathsf{Q} \in \mathcal{P} : \mathsf{Q}(A) \le \Psi_\mu(\mathsf{P}(A)) \text{ for any } A \in \mathcal{F}\} \\ &= \Big\{Z \in L^0 : Z \ge 0,\ \mathsf{E}_\mathsf{P} Z = 1,\ \int_{1-x}^1 q_s(Z)ds \le \Psi_\mu(x)\ \forall x \in [0,1]\Big\},\end{aligned}$$

where q_s is the s-quantile and Ψ_μ is given by (3.4).

For the needs of Sections 7 and 8, we will now provide one more representation of $\mathcal{D}_\mu$. Let us consider the conjugate to the function Ψ_μ:

$$\Phi_\mu(x) = \sup_{y\in[0,1]} [\Psi_\mu(y) - xy], \quad x \in \mathbb{R}_+.$$

Clearly, Φ_μ is decreasing, convex, and has the properties

$$\begin{aligned}&\Phi_\mu(x) = 1 - x, \quad x \le \mu(\{1\}),\\ &\Phi_\mu(x) > 1 - x, \quad x > \mu(\{1\}),\\ &\Phi_\mu(x) > \mu(\{0\}), \quad x < \int_{(0,1]} \lambda^{-1}\mu(d\lambda),\\ &\Phi_\mu(x) = \mu(\{0\}), \quad x \ge \int_{(0,1]} \lambda^{-1}\mu(d\lambda),\\ &\lim_{x\to\infty} \Phi_\mu(x) = \mu(\{0\}).\end{aligned}$$

Furthermore,

$$\Psi_\mu(x) = \inf_{y\in\mathbb{R}_+} [\Phi_\mu(y) + xy], \quad x \in (0,1].$$

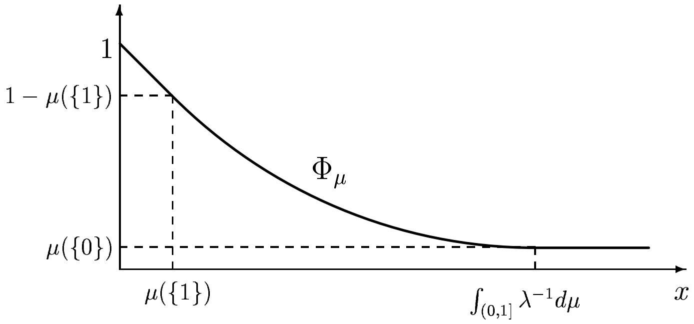

Figure 3. The structure of Φ_μ

Theorem 4.6. *We have*

$$\mathcal{D}_\mu = \{Z \in L^0 : Z \geq 0,\ \mathsf{E_P} Z = 1,\ \text{and } \mathsf{E_P}(Z-x)^+ \leq \Phi_\mu(x)\ \forall x \in [0,1]\}. \tag{4.2}$$

`Proof`. Let $Z \in \mathcal{D}_\mu$. Take $x \in \mathbb{R}_+$ and set $y = \mathsf{P}(Z > x)$. Using Theorem 4.5, we get

$$\mathsf{E_P}(Z-x)^+ = \int_{1-y}^1 q_s(Z)ds - xy \leq \Psi_\mu(y) - xy \leq \Phi_\mu(x).$$

Now, let Z belong to the right-hand side of (4.2). Take $x > 0$ and set $y = q_{1-x}(Z)$. For $y' > y$, we have

$$\mathsf{E_P}(Z-y')^+ - \mathsf{E_P}(Z-y)^+ \geq (y-y')\mathsf{P}(Z > y) \geq (y-y')x,$$

while for $y' < y$, we have

$$\mathsf{E_P}(Z-y')^+ - \mathsf{E_P}(Z-y)^+ \geq (y-y')\mathsf{P}(Z \geq y) \geq (y-y')x.$$

Thus,

$$\begin{aligned}\int_{1-x}^1 q_s(Z)ds &= \mathsf{E_P}(Z-y)^+ + xy = \inf_{y' \in \mathbb{R}_+}[\mathsf{E_P}(Z-y')^+ + xy'] \\ &\leq \inf_{y' \in \mathbb{R}_+}[\Phi_\mu(y') + xy'] = \Psi_\mu(x).\end{aligned}$$

By Theorem 4.5, $Z \in \mathcal{D}_\mu$. □

5 Strict Diversification and Optimization

Let us introduce the notation

$$L^1_\mu = \{X \in L^0 : u_\mu(X) > -\infty \text{ and } u_\mu(-X) > -\infty\}.$$

For more information on the L^1-spaces related to coherent risk measures, see [16; Subsect. 2.2].

Theorem 5.1. *Suppose that* $\operatorname{supp}\mu = [0,1]$. *For* $X, Y \in L^1_\mu$, *we have*

$$u_\mu(X+Y) > u_\mu(X) + u_\mu(Y) \tag{5.1}$$

if and only if X *and* Y *are not comonotone.*

`Proof`. The "only if" part for bounded X and Y is a consequence of Theorem 2.9 (i). The statement for unbounded X and Y is obtained by passing on to the limit with the help of the representation

$$L^1_\mu = \Big\{X \in L^0 : \lim_{n\to\infty} \sup_{\mathsf{Q}\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{Q}|X|I(|X|>n)=0\Big\},$$

which was proved in [16; Subsect. 2.2].

Let us prove the "if" part. Suppose that (5.1) is not true. Combining representation (3.2) with the property $u_\lambda(X+Y) \ge u_\lambda(X)+u_\lambda(Y)$, we conclude that $u_\lambda(X+Y)=u_\lambda(X)+u_\lambda(Y)$ for μ-a.e. $\lambda \in [0,1]$. As $\operatorname{supp}\mu=[0,1]$ and the functions u_λ are continuous in λ, we get $u_\lambda(X+Y)=u_\lambda(X)+u_\lambda(Y)$ for any $\lambda\in[0,1]$. In view of Proposition 2.6 (i), for any $\lambda\in(0,1]$, there exists $Z^*_\lambda \in \mathcal{D}_\lambda$ such that $\mathsf{E}_\mathsf{P} XZ^*_\lambda = u_\lambda(X)$ and $\mathsf{E}_\mathsf{P} YZ^*_\lambda = u_\lambda(Y)$. It is seen from Proposition 2.6 (i) that this is possible only if

$$\begin{aligned} &\mathsf{P}(X<q_\lambda(X),\, Y>q_\lambda(Y))=0, \quad \lambda\in(0,1],\\ &\mathsf{P}(X>q_\lambda(X),\, Y<q_\lambda(Y))=0, \quad \lambda\in(0,1]. \end{aligned}$$

From this it is easy to deduce that $\mathsf{P}((X,Y) \in f((0,1]))=1$, where $f(\lambda)=(q_\lambda(X),q_\lambda(Y))$. Thus, X and Y are comonotone. □

`Remark`. Without the condition $\operatorname{supp}\mu=[0,1]$, the theorem does not hold. In particular, it does not hold for Tail V@R. Let us remark that the problem to provide a corresponding example as well as the problem to prove (5.1) for independent X and Y were proposed at the Fourth Kolmogorov Students' Competition on Probability Theory (see [15]).

Property (5.1) can be called the *strict diversification property*. It holds, in particular, if X and Y are independent or if X and Y have a joint density (with respect to the Lebesgue measure). The strict diversification property leads to the uniqueness of a solution of several optimization problems based on coherent risk measures that were considered in [17]. Let us briefly describe two of them.

Let $S_0\in\mathbb{R}^d$ be the vector of initial prices of several assets and S_1 be the d-dimensional random vector of their terminal discounted prices. Let $H\subseteq\mathbb{R}^d$ be a convex cone of possible trading strategies, so that the discounted cash flow of a strategy $h\in H$ is $\langle h, S_1-S_0\rangle$. The problem is

$$\begin{cases} \mathsf{E}_\mathsf{P}\langle h,S_1-S_0\rangle \longrightarrow \max,\\ h\in H,\ \rho(\langle h,S_1-S_0\rangle)\le c, \end{cases} \tag{5.2}$$

where ρ is a coherent risk measure and $c\ge 0$ (this problem is a particular case of (1.4)). In [17; Subsect. 2.2], we presented a geometric solution of this problem. Here we give a sufficient condition for the uniqueness.

Corollary 5.2. *Let* $u=u_\mu$ *with* $\operatorname{supp}\mu=[0,1]$. *Suppose that each component of* S_1 *belongs to* L^1_μ, S_1 *has a density with respect to the Lebesgue measure, and* $\sup \mathsf{E}_\mathsf{P}\langle h,S_1-S_0\rangle<\infty$, *where* sup *is taken over* $h\in H$ *such that* $\rho(\langle h,S_1-S_0\rangle)\le c$. *Then a solution of* (5.2) (*if it exists*) *is unique.*

`Proof`. Let h_*, h'_* be different solutions. By Theorem 5.1, $h=(h_*+h'_*)/2$ satisfies $\rho(\langle h,S_1-S_0\rangle)<c$. Taking $(1+\varepsilon)h$ with a small $\varepsilon>0$, we get a strategy that performs better than h_*, h'_*. □

Remark. Without the assumption $\operatorname{supp}\mu = [0,1]$, the statement above is not true (see [17; Subsect. 2.2]).

Consider now a single-agent optimization problem. Thus, in addition to the objects introduced above, we have a random variable W, which means the current endowment of some agent. Consider the problem

$$u(W + \langle h, S_1 - S_0\rangle) \xrightarrow[h\in H]{} \max. \tag{5.3}$$

In [17; Subsect. 2.5], we gave a geometric solution of this problem. Theorem 5.1 yields

Corollary 5.3. *Let $u = u_\mu$ with $\operatorname{supp}\mu = [0,1]$. Suppose that each component of S_1 belongs to L^1_μ, S_1 has a density with respect to the Lebesgue measure, $W \in L^1_\mu$, and $\sup_{h\in H} u_\mu(W + \langle h, S_1 - S_0\rangle) < \infty$. Then a solution of (5.3) (if it exists) is unique.*

6 Minimal Extreme Measure and Capital Allocation

The following definition was introduced in [16].

Definition 6.1. Let u be a coherent utility function on L^0 with the determining set $\mathcal{D}$. Let $X \in L^0$. We call a measure $\mathsf{Q} \in \mathcal{D}$ an *extreme measure* for X if $\mathsf{E_Q}X = u(X) \in (-\infty, \infty)$.

The set of extreme measures for $u = u_\mu$ will be denoted by $\mathcal{X}_\mu(X)$.

It is seen from Theorem 3.2 (i) that $\mathcal{X}_\mu(X) \neq \emptyset$ provided that $\mu(\{0\}) = 0$ and $X \in L^1_\mu$.

Proposition 6.2. *Suppose that $\mu(\{0\}) = 0$ and $X \in L^1_\mu$. Then an element $Z = \int_{(0,1]} Z_\lambda \mu(d\lambda) \in \mathcal{D}_\mu$ (here we use the representation of $\mathcal{D}_\mu$ provided by Theorem 4.4) belongs to $\mathcal{X}_\mu(X)$ if and only if*

$$Z_\lambda = \begin{cases} \lambda^{-1} & \text{a.e. on } \{X < q_\lambda(X)\},\\ 0 & \text{a.e. on } \{X > q_\lambda(X)\}\end{cases}$$

for μ-a.e. λ.

Proof. For $Z = \int_{(0,1]} Z_\lambda \mu(d\lambda) \in \mathcal{D}$, we have

$$\begin{aligned} \mathsf{E_P}XZ &= \lim_{n\to\infty}\lim_{m\to-\infty} \mathsf{E_P}(m \vee X \wedge n)Z\\ &= \lim_{n\to\infty}\lim_{m\to-\infty} \int_{(0,1]} [\mathsf{E_P}(m\vee X\wedge n)Z_\lambda]\mu(d\lambda)\\ &= \int_{(0,1]} (\mathsf{E_P}XZ_\lambda)\mu(d\lambda). \end{aligned} \tag{6.1}$$

The inclusion $X \in L^1_\mu$ implies that the function $\lambda \mapsto \mathsf{E_P}XZ_\lambda$ is μ-integrable. An application of Proposition 2.6 (i) completes the proof. □

It is seen from the above proposition that if X has a continuous distribution, then $\mathcal{X}_\mu(X)$ consists of a unique element $Z = g(X)$, where

$$g(x) = \int_{[F(x),1]} \lambda^{-1}\mu(d\lambda), \quad x \in \mathbb{R} \tag{6.2}$$

and F denotes the distribution function of X.

If $\operatorname{Law} X$ has atoms, then, clearly, $\mathcal{X}_\mu(X)$ need not be a singleton. However, it turns out that there exists a minimal element of $\mathcal{X}_\mu(X)$ with respect to the convex stochastic order. (For other applications of this order in financial mathematics, see [30], [34], [35].)

Theorem 6.3. *Suppose that* $\mu(\{0\}) = 0$ *and* $X \in L^1_\mu$. *Let*

$$Z^*_\mu(X) = \int_{(0,1]} Z^*_\lambda(X)\mu(d\lambda),$$

where $Z^*_\lambda(X)$ *is defined by* (3.1). *Then, for any* $Z \in \mathcal{X}_\mu(X)$ *and any convex function* $f : \mathbb{R}_+ \to \mathbb{R}_+$, *we have* $\mathsf{E}_\mathsf{P} f(Z^*_\mu(X)) \le \mathsf{E}_\mathsf{P} f(Z)$. *Moreover,* $Z^*_\mu(X)$ *is the unique element of* $\mathcal{X}_\mu(X)$ *with this property.*

`Proof`. Take an arbitrary $Z = \int_{(0,1]} Z_\lambda \mu(d\lambda) \in \mathcal{X}_\mu(X)$. It follows from Proposition 6.2 that $Z^*_\lambda(X) = \mathsf{E}_\mathsf{P}(Z_\lambda \mid X)$ for μ-a.e. λ. By Fubini's theorem, $Z^*_\mu(X) = \mathsf{E}_\mathsf{P}(Z \mid X)$. An application of Jensen's inequality yields the first statement.

Now, suppose that there exists another minimal (in the convex order) element Z' of $\mathcal{X}_\mu(X)$. Then $\widetilde{Z} := (Z^*_\mu(X) + Z')/2$ belongs to $\mathcal{X}_\mu(X)$ and, for a strictly convex function f with a linear growth, we get $\mathsf{E}_\mathsf{P} f(\widetilde{Z}) < \mathsf{E}_\mathsf{P} f(Z^*_\mu(X))$, which is a contradiction. □

Definition 6.4. We will call the measure $\mathsf{Q}^*_\mu(X) = Z^*_\mu(X)\mathsf{P}$ the *minimal extreme measure* for X.

The minimal extreme measure admits a representation similar to (6.2). Let F denote the distribution function of X. Then, for any $\lambda \in (0,1]$, $Z^*_\lambda(X) = g_\lambda(X)$, where

$$g_\lambda(x) = \frac{1 - \lambda^{-1}F(x-)}{F(x) - F(x-)} I(F(x-) < \lambda < F(x)) + \lambda^{-1} I(\lambda \ge F(x)).$$

Hence, $Z^*_\mu(X) = g(X)$, where

$$g(x) = \int_{(F(x-),F(x))} \frac{1 - \lambda^{-1}F(x-)}{F(x) - F(x-)} \mu(d\lambda) + \int_{[F(x),1]} \lambda^{-1}\mu(d\lambda).$$

The following statement will be used in financial applications below.

Theorem 6.5. *Suppose that* $\mu(\{0\}) = 0$ *and* $X, Y \in L^1_\mu$. *Let* (ξ_n) *be a sequence of random variables such that* $\xi_n \in L^1_\mu$, *each* ξ_n *is independent of* (X, Y), *and* $\xi_n \xrightarrow{\mathsf{P}} 0$. *Then*

$$\mathsf{E}_{\mathsf{Q}^*_\mu(X+\xi_n)} Y \xrightarrow[n\to\infty]{} \mathsf{E}_{\mathsf{Q}^*_\mu(X)} Y.$$

`Proof`. Denote $q_\lambda = q_\lambda(X)$, $q^n_\lambda = q_\lambda(X + \xi_n)$. Then, for $\lambda \in (0,1]$,

$$Z^*_\lambda(X) = \begin{cases} \lambda^{-1} & \text{on } \{X < q_\lambda\}, \\ c_\lambda & \text{on } \{X = q_\lambda\}, \\ 0 & \text{on } \{X > q_\lambda\}, \end{cases} \quad Z^*_\lambda(X + \xi_n) = \begin{cases} \lambda^{-1} & \text{on } \{X + \xi_n < q^n_\lambda\}, \\ c^n_\lambda & \text{on } \{X + \xi_n = q^n_\lambda\}, \\ 0 & \text{on } \{X + \xi_n > q^n_\lambda\}. \end{cases}$$

Fix $\lambda \in (0,1]$. By Fubini's theorem, $\mathsf{E}_\mathsf{P}(Z^*_\lambda(X + \xi_n) \mid X, Y) = f^n_\lambda(X)$, where

$$f^n_\lambda(x) = \lambda^{-1}F_n(q^n_\lambda - x) + c^n_\lambda \Delta F_n(q^n_\lambda - x), \quad x \in \mathbb{R}$$

and $F_n(x) = \mathsf{P}(\xi_n < x)$, $\Delta F_n(x) = \mathsf{P}(\xi_n = x)$. Obviously, $q_\lambda^n \to q_\lambda$, and therefore, $f_\lambda^n \to \lambda^{-1}$ on $(-\infty, q_\lambda)$, $f_\lambda^n \to 0$ on (q_λ, ∞). Employing the normalization condition $\mathsf{E_P} f_\lambda^n(X) = 1$, we conclude that $f_\lambda^n(q_\lambda) \to c_\lambda$. Thus, $f_\lambda^n(X) \xrightarrow{\text{a.s.}} Z_\lambda^*(X)$. As $0 \le f_\lambda^n \le \lambda^{-1}$, we get

$$\mathsf{E_P} Y Z_\lambda^*(X + \xi_n) = \mathsf{E_P} Y f_\lambda^n(X) \xrightarrow[n\to\infty]{} \mathsf{E_P} Y Z_\lambda^*(X), \quad \lambda \in (0,1].$$

Note that $u_\lambda(Y) \le \mathsf{E_P} Y Z_\lambda^*(X + \xi_n) \le -u_\lambda(-Y)$. Furthermore, it follows from the inclusion $Y \in L_\mu^1$ and representation (3.2) that the functions $\lambda \mapsto u_\lambda(Y)$ and $\lambda \mapsto u_\lambda(-Y)$ are μ-integrable. Applying now (6.1), we complete the proof. □

Remark. Without the assumption that ξ_n is independent of (X, Y), the theorem does not hold. As an example, consider $X = 0$, $\xi_n = Y/n$. Then $\mathsf{Q}_\mu^*(X) = \mathsf{P}$, while $\mathsf{Q}_\mu^*(\xi_n) = \mathsf{Q}_\mu^*(Y)$.

Let us now present a financial application of the notion of the minimal extreme measure. It is related to the *capital allocation problem*. Delbaen [21; Sect. 9] proposed the following formulation of this problem. Let $X^1, \dots, X^d$ be random variables meaning the discounted cash flows produced by several components of a firm. Let ρ be a coherent risk measure. A *capital allocation between* $X^1, \dots, X^d$ is a vector $x^1, \dots, x^d$ such that

$$\rho\Big(\sum_{i=1}^d X^i\Big) = \sum_{i=1}^d x^i, \tag{6.3}$$

$$\forall h^1, \dots, h^d \in \mathbb{R}_+, \quad \rho\Big(\sum_{i=1}^d h^i X^i\Big) \ge \sum_{i=1}^d h^i x^i. \tag{6.4}$$

From the financial point of view, x^i means the contribution of the i-th component to the total risk of the firm, or, equivalently, the capital that should be allocated to this component. In order to illustrate the meaning of (6.4), consider the example $h^i = I(i \in J)$, where J is a subset of $\{1, \dots, d\}$. Then (6.4) means that the capital allocated to a part of the firm does not exceed the risk carried by that part.

It was proved in [16; Subsect. 2.4] under the assumption $u(X^i) > -\infty$, $u(-X^i) > -\infty$, $i = 1, \dots, d$ (here $u = -\rho$) that the set of capital allocations has the form

$$\Big\{-\mathsf{E_Q}(X^1, \dots, X^d) : \mathsf{Q} \in \mathcal{X}_\rho\Big(\sum_{i=1}^d X^i\Big)\Big\}, \tag{6.5}$$

where $\mathcal{X}_\rho$ denotes the set of extreme measures corresponding to ρ.

Suppose now that $u = u_\mu$ with $\mu(\{0\}) = 0$. It is seen from Proposition 6.1 that if $\sum_i X^i$ has a continuous distribution, then a capital allocation is unique. But in general, this is not the case. For example, if $X^2 = -X^1$, then the set of capital allocations is the interval $[a, b]$ in $\mathbb{R}^2$, where $a = (-u_\mu(X^1), u_\mu(X^1))$, $b = (u_\mu(-X^1), -u_\mu(-X^1))$.

However, if $u = u_\mu$ with $\mu(\{0\}) = 0$, then there exists a particular element of (6.5), namely $x_0 = -\mathsf{E}_{\mathsf{Q}_\mu^*(\sum X^i)}(X^1, \dots, X^d)$ (for the example considered above, $x_0 = (-\mathsf{E_P} X^1, \mathsf{E_P} X^1)$). We call x_0 the *central solution* of the capital allocation problem. Its role is as follows. Let us disturb X^i, i.e. we pass from X^i to $\widetilde{X}_n^i = X^i + \xi_n^i$, where each ξ_n^i is independent of $(X^1, \dots, X^d)$, $\xi_n^i \in L_\mu^1$, and $\xi_n^i \xrightarrow{\mathsf{P}} 0$. If $\sum_i \xi_n^i$ has a continuous distribution, then $\mathcal{X}_\mu\big(\sum_i \widetilde{X}_n^i\big)$ is a singleton, so that the capital allocation x_n between $\widetilde{X}_n^1, \dots, \widetilde{X}_n^d$ is unique. By Theorem 6.5, $x_n \to x_0$.

7 Pricing in an Option-Based Model

Let $S_0 \in (0, \infty)$ be the initial price of some asset and S_1 be a positive random variable meaning its terminal discounted price. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices of traded European call options on this asset with maturity 1 and let $\varphi(K)$, $K \in \mathbb{K}$ be the price at time 0 of the option that pays $(S_1 - K)^+$ at time 1. The set

$$A = \left\{ \sum_{n=1}^{N} h_n[(S_1 - K_n)^+ - \varphi(K_n)] : N \in \mathbb{N},\ K_n \in \mathbb{K},\ h_n \in \mathbb{R} \right\}$$

is the set of cash flows that can be obtained in the model under consideration (we assume that $0 \in \mathbb{K}$, which corresponds to the possibility of trading the underlying asset). We also fix a coherent utility function u.

According to [16], we say that the model satisfies the *No Good Deals* (*NGD*) condition if there exists no $X \in A$ with $u(X) > 0$.

Now, let $F \in L^0$ be the payoff of some contingent claim. According to [16], we say that a real number z is an *NGD price* of F if there exist no $X \in A$, $h \in \mathbb{R}$ with $u(X + h(F - z)) > 0$. The set of NGD prices will be denoted by $I_{NGD}(F)$.

It was proved in [16; Subsect. 3.1] (under some additional conditions that are automatically satisfied for $u = u_\mu$ provided that $\mu(\{0\}) = 0$) that the NGD condition is satisfied if and only if $\mathcal{D} \cap \mathcal{R} \neq \emptyset$, where $\mathcal{D}$ is the determining set of u and $\mathcal{R}$ is the set of *risk-neutral measures*, which in this model has the form

$$\mathcal{R} = \{ \mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q}(S_1 - K)^+ = \varphi(K) \text{ for any } K \in \mathbb{K} \}$$

(the notation $\mathcal{P}$ was introduced in Section 2). Furthermore, for $F \in L^0$ such that $u(F) > -\infty$ and $u(-F) > -\infty$,

$$I_{NGD}(F) = \{ \mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R} \}. \tag{7.1}$$

Below we give more concrete versions of these results for $u = u_\mu$. Let us introduce the notation

$$\begin{aligned} \mathfrak{F}_\mu = \Big\{ \psi : \ & \psi \text{ is a convex function } \mathbb{R}_+ \to \mathbb{R}_+,\ \psi'_+(0) \geq -1, \\ & \lim_{x \to \infty} \psi(x) = 0,\ \psi|_\mathbb{K} = \varphi|_\mathbb{K},\ \psi'' \sim \mathsf{P}_0, \text{ and} \\ & \int_{\mathbb{R}_+} \Big(\frac{d\psi''}{d\mathsf{P}_0}(y) - x \Big)^+ \mathsf{P}_0(dy) \leq \Phi_\mu(x) \text{ for any } x \in \mathbb{R}_+ \Big\}. \end{aligned}$$

Here ψ'_+ denotes the right-hand derivative, ψ'' is the second derivative taken in the sense of distributions (i.e. $\psi''((a,b]) = \psi'_+(b) - \psi'_+(a)$) with the convention $\psi''(\{0\}) = \psi'_+(0) + 1$, $\mathsf{P}_0 = \mathrm{Law}_\mathsf{P}\, S_1$, and Φ_μ is the function introduced in Section 4.

Theorem 7.1. *Let $u = u_\mu$ with $\mu(\{0\}) = 0$.*
(i) *The NGD is satisfied if and only if $\mathfrak{F}_\mu \neq \emptyset$.*
(ii) *For $F = f(S_1) \in L^1_\mu$, we have*

$$I_{NGD}(F) = \left\{ \int_{\mathbb{R}_+} f(x) \psi''(dx) : \psi \in \mathfrak{F}_\mu \right\}.$$

Proof. **(i)** Let us prove the "only if" part. By the result mentioned above, there exists $\mathsf{Q} \in \mathcal{D}_\mu \cap \mathcal{R}$. The function $\psi(x) := \mathsf{E}_\mathsf{Q}(S_1 - x)^+$ is convex, $\lim_{x\to\infty} \psi(x) = 0$, and $\psi|_\mathbb{K} = \varphi|_\mathbb{K}$. Denote $Z = \frac{d\mathsf{Q}}{d\mathsf{P}}$ and set $g(x) = \mathsf{E}_\mathsf{P}(Z \mid S_1 = x)$. Then

$$\psi(x) = \mathsf{E}_\mathsf{P}(S_1 - x)^+ g(S_1) = \int_{\mathbb{R}_+} (y - x)^+ g(y) \mathsf{P}_0(dy), \quad x \in \mathbb{R}.$$

This representation shows that $\psi'_+(0) \geq -1$ and $\psi'' = g\mathsf{P}_0$. By Theorem 4.6,

$$\int_{\mathbb{R}_+} (g(y) - x)^+ \mathsf{P}_0(dy) = \mathsf{E}_\mathsf{P}(g(S_1) - x)^+ \leq \mathsf{E}_\mathsf{P}(Z - x)^+ \leq \Phi_\mu(x), \quad x \in \mathbb{R},$$

so that $\psi \in \mathfrak{F}_\mu$.

Let us prove the "if" part. Take $\psi \in \mathfrak{F}_\mu$ and set $g = \frac{d\psi''}{d\mathsf{P}_0}$, $\mathsf{Q} = g(S_1)\mathsf{P}$. Then $\mathsf{Q} \in \mathcal{P}$. The inequality

$$\mathsf{E}_\mathsf{Q}(g(S_1) - x)^+ = \int_{\mathbb{R}_+} (g(y) - x)^+ \mathsf{P}_0(dy) \leq \Phi_\mu(x), \quad x \in \mathbb{R}_+,$$

combined with Theorem 4.6, shows that $\mathsf{Q} \in \mathcal{D}_\mu$. Furthermore,

$$\mathsf{E}_\mathsf{Q}(S_1 - K)^+ = \int_{\mathbb{R}_+} (y - K)^+ \psi''(dy) = \psi(K) = \varphi(K), \quad K \in \mathbb{K},$$

so that $\mathsf{Q} \in \mathcal{R}$.

(ii) Take $z \in I_{NGD}(F)$. By (7.1), $z = \mathsf{E}_\mathsf{Q} f(S_1)$ with some $\mathsf{Q} \in \mathcal{D}_\mu \cap \mathcal{R}$. The proof of (i) shows that

$$z = \mathsf{E}_\mathsf{P} f(S_1) g(S_1) = \int_{\mathbb{R}_+} f(x) g(x) \mathsf{P}_0(dx) = \int_{\mathbb{R}_+} f(x) \psi''(dx),$$

where $g(x) = \mathsf{E}_\mathsf{P}\big(\frac{d\mathsf{Q}}{d\mathsf{P}} \mid S_1 = x\big)$ and $\psi = \mathsf{E}_\mathsf{Q}(S_1 - \cdot)^+ \in \mathfrak{F}_\mu$.

Conversely, take $z = \int_{\mathbb{R}_+} f(x)\psi''(dx)$ with $\psi \in \mathfrak{F}_\mu$. Then $z = \mathsf{E}_\mathsf{Q} f(S_1)$, where $\mathsf{Q} = g(S_1)\mathsf{P}$, $g = \frac{d\psi''}{d\mathsf{P}_0}$. The proof of (i) shows that $\mathsf{Q} \in \mathcal{D}_\mu \cap \mathcal{R}$, and by (7.1), $z \in I_{NGD}(F)$. □

8 Optimization in a Complete Model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. We consider a complete model, in which an agent can obtain by trading any cash flow from the class

$$A = \{X \in L^1(\mathsf{Q}) : \mathsf{E}_\mathsf{Q} X = 0\},$$

where Q is a fixed probability measure, which is absolutely continuous with respect to P. Clearly, problem (1.4) is equivalent to the problem

$$\text{RAROC}(X) \xrightarrow[X \in A]{} \max, \tag{8.1}$$

where

$$\text{RAROC}(X) = \begin{cases} +\infty & \text{if } \mathsf{E}_\mathsf{P} X > 0 \text{ and } u(X) \geq 0, \\ \dfrac{\mathsf{E}_\mathsf{P} X}{-u(X)} & \text{otherwise.} \end{cases}$$

We will take $u = u_\mu$, where $\mu \neq \delta_1$ (otherwise $u_\mu(X) = \mathsf{E}_\mathsf{P} X$, and the above problem is meaningless).

Let Φ_μ be the function defined in Section 4 and set

$$\varphi(x) = \mathsf{E}_\mathsf{P}(Z^0 - x)^+, \quad x \in \mathbb{R}_+,$$

where $Z^0 = \frac{d\mathsf{Q}}{d\mathsf{P}}$. We will assume that there exists no $X \in A$ with $u_\mu(X) > 0$ (indeed, for such X we would have $\mathrm{RAROC}(X) = \infty$). In view of the results of [16; Subsect. 3.2], this is equivalent to the inclusion $\mathsf{Q} \in \mathcal{D}_\mu$. This, in turn, is equivalent to the inequality $\varphi \leq \Phi_\mu$ (see Theorem 4.6). For $\alpha \geq 1$, we set

$$\varphi_\alpha(x) = \alpha\varphi\Big(\frac{x-1}{\alpha} + 1\Big), \quad x \in \mathbb{R}_+,$$

i.e. the graph of φ_α is the α-stretching of the graph of φ with respect to the point $(1,0)$ (see Figure 4). Set $\alpha_* = \sup\{\alpha : \varphi_\alpha \leq \Phi_\mu\}$, $\beta = \mu(\{1\})$, $\gamma = \int_{(0,1]} \lambda^{-1}\mu(d\lambda)$, and $\overline{Z} = \alpha_*\big(Z^0 - \frac{\alpha_*-1}{\alpha_*}\big)$, so that $\varphi_{\alpha_*}(x) = \mathsf{E}_\mathsf{P}(\overline{Z} - x)^+$. Note that the left-hand and the right-hand derivatives of φ_{α_*} are given by

$$(\varphi_{\alpha_*})'_-(x) = -\mathsf{P}(\overline{Z} \geq x), \quad x > 0, \tag{8.2}$$

$$(\varphi_{\alpha_*})'_+(x) = -\mathsf{P}(\overline{Z} > x), \quad x \geq 0. \tag{8.3}$$

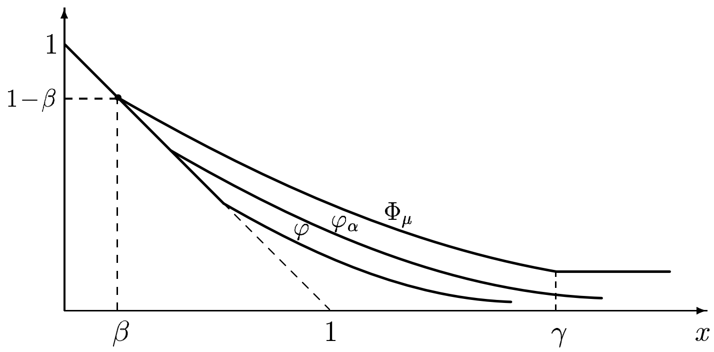

Figure 4. The structure of Φ_μ, φ, and φ_α

Theorem 8.1. *Suppose that* $\mu \neq \delta_1$, $\mathsf{Q} \neq \mathsf{P}$, *and* $\varphi \leq \Phi_\mu$.

(i) *We have* $\sup_{X \in A} \mathrm{RAROC}(X) = (\alpha_* - 1)^{-1}$.

(ii) *If* $\alpha_* > 1$ *and* $(\varphi_{\alpha_*})'_+(\beta) > -1$, *then* $\mathsf{P}(\overline{Z} = \beta) > 0$ *and any random variable of the form* $X = bI_B + cI_{B^c}$, *where* $B \subseteq \{\overline{Z} = \beta\}$ *and* $b > 0 > c$ *are such that* $\mathsf{E}_\mathsf{Q} X = 0$, *is optimal for* (8.1).

(iii) *If* $\alpha_* > 1$, $\gamma < \infty$, *and either* $\varphi_{\alpha_*}(\gamma) = \Phi_\mu(\gamma) > 0$ *or* $\Phi_\mu(\gamma) = 0$ *and* $(\varphi_{\alpha_*})'_-(\gamma) < 0$, *then* $\mathsf{P}(\overline{Z} \geq \gamma) > 0$ *and any random variable of the form* $X = bI_B + cI_{B^c}$, *where* $B \subseteq \{\overline{Z} \geq \gamma\}$ *and* $b < 0 < c$ *are such that* $\mathsf{E}_\mathsf{Q} X = 0$, *is optimal for* (8.1).

(iv) *If* $\alpha_* > 1$ *and there exists* $x_0 \in (\beta, \gamma)$ *such that* $\varphi_{\alpha_*}(x_0) = \Phi_\mu(x_0)$, *then* $\mathsf{P}(\overline{Z} > x_0) > 0$ *and any random variable of the form* $X = bI_B + cI_{B^c}$, *where* $B = \{\overline{Z} > x_0\}$ *and* $b < 0 < c$ *are such that* $\mathsf{E}_\mathsf{Q} X = 0$, *is optimal for* (8.1). *If moreover* $\operatorname{supp}\mu = [0,1]$ *and* x_0 *is the unique point of* (β, γ), *at which* $\varphi_{\alpha_*} = \Phi_\mu$, *then an optimal element of* A *is unique up to multiplication by a positive constant.*

(v) *If* $\alpha_* > 1$, *but neither of conditions* (ii)–(iv) *is satisfied, then the maximum in* (8.1) *is not attained.*

Remarks. (i) It is easy to check that if μ has no gap near 1, i.e. $\mu((1-\varepsilon,1))>0$ for any $\varepsilon>0$, then $(\Phi_\mu)'_+(\beta)=-1$. Similarly, if μ has no gap near 0, i.e. $\mu((0,\varepsilon))>0$ for any $\varepsilon>0$, then $(\Phi_\mu)'_-(\gamma)=0$. Thus, the situation of (ii) (resp., (iii)) can be realized only if μ has a gap near 1 (resp., near 0). Another verification of these statements follows from the arguments in the proof of (v) below.

(ii) In many natural complete models (for instance, in the Black-Scholes model), we have $\operatorname{essinf}_\omega Z^0(\omega)=0$. Then $\varphi(\varepsilon)>1-\varepsilon$ for any $\varepsilon>0$, and hence, $\alpha_*=1$. By Theorem 8.1 (i), $\sup_{X\in A}\mathrm{RAROC}(X)=\infty$. A sequence of elements $X_n\in A$ with $\mathrm{RAROC}(X_n)\to\infty$ is provided by $X_n=I(Z\le n^{-1})-\mathsf{Q}(Z\le n^{-1})$.

Proof of Theorem 8.1 **(i)** Take $R\in(0,\infty)$. It follows from the result in [16; Subsect. 3.2] that $\sup_{X\in A}\mathrm{RAROC}(X)\le R$ if and only if

$$\frac{1+R}{R}\left(Z^0-\frac{1}{1+R}\right)\in\mathcal{D}_\mu.$$

In view of Theorem 4.6, this is equivalent to the conditions $Z^0\ge\frac{1}{1+R}$ and

$$\forall x\ge 0,\ \ \mathsf{E_P}\left(\frac{1+R}{R}\left(Z^0-\frac{1}{1+R}\right)-x\right)^+\le\Phi_\mu(x). \tag{8.4}$$

Note that

$$\mathsf{E_P}\left(\frac{1+R}{R}\left(Z^0-\frac{1}{1+R}\right)-x\right)^+=\varphi_{\alpha(R)}(x),\ \ x\in\mathbb{R}_+,$$

where $\alpha(R)=\frac{1+R}{R}$. Thus, (8.4) is satisfied if and only if $\varphi_{\alpha(R)}\le\Phi_\mu$.

Set $h=\operatorname{essinf}_\omega Z^0(\omega)$. Note that $\varphi(h)=1-h$ and $\varphi(h+\varepsilon)>1-h-\varepsilon$ for any $\varepsilon>0$. As $\Phi_\mu(0)=1$, we conclude that the condition $\varphi_{\alpha(R)}\le\Phi_\mu$ automatically implies that $(1-h)\alpha(R)\le 1$, which, in turn, is equivalent to $Z^0\ge\frac{1}{1+R}$. As a result,

$$\sup_{X\in A}\mathrm{RAROC}(X)\le R\Longleftrightarrow\varphi_{\alpha(R)}\le\Phi_\mu\Longleftrightarrow\alpha(R)\le\alpha_*\Longleftrightarrow R\ge(\alpha_*-1)^{-1}.$$

(ii) The inequality

$$\mathsf{E_P}(\overline{Z}-x)^+=\varphi_{\alpha_*}(x)\le\Phi_\mu(x),\quad x\in\mathbb{R}_+, \tag{8.5}$$

combined with Theorem 4.6, shows that $\overline{Z}\in\mathcal{D}_\mu$. Consequently, $\overline{Z}\ge\beta$, and it follows from (8.3) that $\mathsf{P}(\overline{Z}=\beta)>0$. Due to Theorem 4.4, we can write $\overline{Z}=\int_{[0,1]}\overline{Z}_\lambda\mu(d\lambda)$ with $\overline{Z}_\lambda\in\mathcal{D}_\lambda$. As $\overline{Z}_1=1$, we deduce that, for μ-a.e. $\lambda\in[0,1)$, $\overline{Z}_\lambda=0$ P-a.e. on $\{\overline{Z}=\beta\}$. Then, due to the structure of X, for μ-a.e. $\lambda\in[0,1]$, we have $\mathsf{E_P}X\overline{Z}_\lambda=u_\lambda(X)$. Thus, $\mathsf{E_P}X\overline{Z}=u_\mu(X)$. Applying now the equality

$$0=\mathsf{E_P}XZ^0=\frac{\alpha_*-1}{\alpha_*}\mathsf{E_P}X+\frac{1}{\alpha_*}\mathsf{E_P}X\overline{Z}=\frac{\alpha_*-1}{\alpha_*}\mathsf{E_P}X+\frac{1}{\alpha_*}u_\mu(X),$$

we deduce that $\mathrm{RAROC}(X)=(\alpha_*-1)^{-1}$, so that X is optimal.

(iii) Consider first the case $\varphi_{\alpha_*}(\gamma)=\Phi_\mu(\gamma)>0$. Due to (8.5), $\overline{Z}\in\mathcal{D}_\mu$, so that we can write $\overline{Z}=\int_{[0,1]}\overline{Z}_\lambda\mu(d\lambda)=\xi+\eta$, where $\xi=\int_{(0,1]}\overline{Z}_\lambda\mu(d\lambda)$ and $\eta=\mu(\{0\})\overline{Z}_0$. As $\overline{Z}_\lambda\le\lambda^{-1}$, we get $\xi\le\gamma$, so that

$$\varphi_{\alpha_*}(\gamma)=\mathsf{E_P}(\overline{Z}-\gamma)^+\le\mathsf{E_P}\eta=\mu(\{0\})=\Phi_\mu(\gamma).$$

The fact that this inequality should be an equality means that $\mathsf{P}(\xi = \gamma) > 0$ and $\eta = 0$ P-a.e. outside $\{\xi = \gamma\}$. This implies that, for μ-a.e. $\lambda \in (0,1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{\overline{Z} > \gamma\}$ and $\overline{Z}_0 = 0$ P-a.e. outside $\{\overline{Z} > \gamma\}$. The proof is now completed in the same way as in (ii).

Now, consider the case $\Phi_\mu(\gamma) = 0$ and $(\varphi_{\alpha_*})'_-(\gamma) < 0$. As $\Phi_\mu(\gamma) = \mu(\{0\})$, we get $\mu(\{0\}) = 0$. Thus, $\overline{Z} = \int_{(0,1]} \overline{Z}_\lambda \mu(d\lambda)$. As $\overline{Z}_\lambda \le \lambda^{-1}$, we get $\overline{Z} \le \gamma$. It follows from (8.2) that $\mathsf{P}(\overline{Z} = \gamma) > 0$. This implies that, for μ-a.e. $\lambda \in (0,1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{\overline{Z} = \gamma\}$. The proof is now completed in the same way as in (ii).

(iv) Set

$$\lambda_0 = \inf\Big\{x \in [0,1] : \int_{(x,1]} \lambda^{-1}\mu(d\lambda) \le x_0\Big\},$$

$$l = \int_{(\lambda_0,1]} \lambda^{-1}\mu(d\lambda), \quad r = \int_{[\lambda_0,1]} \lambda^{-1}\mu(d\lambda).$$

Consider the case $\mu(\{\lambda_0\}) > 0$ (the other case is simpler). It is easy to see that $x \in [l, r]$ and Φ_μ is linear on $[l, r]$. In view of the convexity of φ_{α_*}, this implies that $\varphi_{\alpha_*} = \Phi_\mu$ on $[l, r]$. As $\int_{(\lambda_0,1]} \overline{Z}_\lambda d\mu \le l$, we get

$$\varphi_{\alpha_*}(l) = \mathsf{E}_\mathsf{P}(\overline{Z} - l)^+ \le \mathsf{E}_\mathsf{P} \int_{[\lambda_0,1]} \overline{Z}_\lambda \mu(d\lambda) = \mu([0,\lambda_0]). \qquad (8.6)$$

Furthermore,

$$\Phi_\mu(l) = \Psi_\mu(\lambda_0) - l\lambda_0 = \int_0^{\lambda_0} (G(x) - l)dx = -\int_{[0,\lambda_0]} x dG(x) = \mu([0,\lambda_0]), \qquad (8.7)$$

where $G(x) = \int_{(x,1]} \lambda^{-1}\mu(d\lambda)$. In a similar way, we check that

$$\varphi_{\alpha_*}(r) = \mathsf{E}_\mathsf{P}(\overline{Z} - r)^+ \le \mathsf{E}_\mathsf{P} \int_{(\lambda_0,1]} \overline{Z}_\lambda \mu(d\lambda) = \mu([0,\lambda_0)) = \Phi_\mu(r). \qquad (8.8)$$

As inequalities in (8.6) and (8.8) should be equalities, we conclude that there exists a set B such that $\overline{Z}_\lambda = \lambda^{-1}$ a.e. on B for $\lambda \ge \lambda_0$ and $\overline{Z}_\lambda = 0$ a.e. on B^c for $\lambda \le \lambda_0$. The proof of the first part of (iv) is now completed in the same way as in (ii).

Let us now prove the uniqueness. Let X be optimal for (8.1). Then X is not degenerate since otherwise it should be equal to 0. Thus, we can find $c \in \mathbb{R}$ such that $\lambda_0 = \mathsf{P}(X \le c)$ belongs to $(0,1)$. The analysis of the proof of (ii) shows that $u_\mu(X) = \mathsf{E}_\mathsf{P} X\overline{Z}$. Consequently, for μ-a.e. $\lambda \in [0,1]$, $u_\lambda(X) = \mathsf{E}_\mathsf{P} X\overline{Z}_\lambda$, where $\overline{Z}_\lambda$ are taken from the representation $\overline{Z} = \int_{[0,1]} \overline{Z}_\lambda \mu(d\lambda)$. This means that, for μ-a.e. $\lambda \in (\lambda_0, 1]$, $\overline{Z}_\lambda = \lambda^{-1}$ P-a.e. on $\{X \le c\}$ and, for μ-a.e. $\lambda \in [0,\lambda_0]$, $\overline{Z}_\lambda = 0$ P-a.e. outside $\{X \le c\}$. Consequently, for $x_0 = \int_{(\lambda_0,1]} \lambda^{-1}\mu(d\lambda)$, we have $\mathsf{E}_\mathsf{P}(\overline{Z} - x_0)^+ = \mu([0,\lambda_0])$. Calculations similar to (8.7) show that $\mu([0,\lambda_0]) = \Phi_\mu(x_0)$. Moreover, as $\lambda_0 \in (0,1)$ and $\operatorname{supp}\mu = [0,1]$, we have $x_0 \in (\beta, \gamma)$. Since such x_0 is unique, we conclude that X takes on only two values. Thus, any optimal X has the form $bI_B + cI_{B^c}$ with some $B \in \mathcal{F}$ and some constants $b < c$. It is clear from the reasoning given above that $\mathsf{P}(B)$ is determined uniquely. Using the same arguments as in the proof of Corollary 5.2, we deduce that different optimal elements should be comonotone. Consequently, B is determined uniquely, so that an optimal strategy is unique up to multiplication by a positive constant.

(v) Assume the contrary, i.e. the existence of an optimal element X. As X is not degenerate, we can find $c \in \mathbb{R}$ such that $\lambda_0 = \mathsf{P}(X \le c)$ belongs to $(0,1)$. Arguing in the same way as above, we prove that $\varphi_{\alpha_*}(x_0)=\Phi_\mu(x_0)$, where $x_0=\int_{(\lambda_0,1]}\lambda^{-1}\mu(d\lambda)$. We will now consider three cases.

Case 1. Assume that $x_0=\beta$. This means that $\mu((\lambda_0,1))=0$. The arguments given above show that, for μ-a.e. $\lambda\in[0,\lambda_0]$, $\overline{Z}=0$ P-a.e. outside $\{X\le c\}$. Consequently, $\overline{Z}=\beta$ P-a.e. on $\{X>c\}$, so that

$$(\varphi_{\alpha_*})'_+(\beta)=-\mathsf{P}(\overline{Z}>\beta)>-1.$$

Thus, in this case conditions of (ii) are satisfied.

Case 2. Assume that $x_0=\gamma$. This means that $\mu((0,\lambda_0))=0$. If $\mu(\{0\})>0$, then $\varphi_{\alpha_*}(\gamma)=\Phi_\mu(\gamma)=\mu(\{0\})>0$, so that conditions of (iii) are satisfied.

Now, assume that $\mu(\{0\})=0$. Then $\Phi_\mu(\gamma)=0$. The arguments given above show that, for μ-a.e. $\lambda\in[\lambda_0,1]$, $\overline{Z}_\lambda=\lambda^{-1}$ P-a.e. on $\{X\le c\}$. As $\mu([0,\lambda_0))=0$, we get $\overline{Z}=\int_{(0,1]}\lambda^{-1}\mu(d\lambda)=\gamma$ P-a.e. on $\{X\le c\}$, so that

$$(\varphi_{\alpha_*})'_-(\gamma)=-\mathsf{P}(\overline{Z}=\gamma)<0.$$

Thus, in this case conditions of (iii) are satisfied.

Case 3. Assume that $x_0\in(\beta,\gamma)$. Then conditions of (iv) are satisfied. □

Corollary 8.2. *Suppose that* $\operatorname{supp}\mu=[0,1]$, $\mathsf{Q}\ne\mathsf{P}$, $\varphi\le\Phi_\mu$, *and* $\alpha_*>1$.

There exists an optimal element of A if and only if there exists $x_0\in(\beta,\gamma)$ such that $\varphi_{\alpha_*}(x_0)=\Phi_\mu(x_0)$.

An optimal element is unique up to multiplication by a positive constant if and only if such x_0 is unique.

`Proof`. The proof of point (v) shows that, under the condition $\operatorname{supp}\mu=[0,1]$, the situations of (ii), (iii) are not realized. Now, the statement follows from Theorem 8.1. □

The financial interpretation of the obtained results is as follows. In most natural situations, the optimal strategy consists in buying the binary option with the payoff

$$I\big(\overline{Z}\le x_0\big)=I\big(Z^0\le\alpha_*^{-1}(x_0-1)+1\big)=I\bigg(\frac{d\mathsf{Q}}{d\mathsf{P}}\le c_*\bigg).$$

The geometric recipe for finding c_* is given in Figure 5.

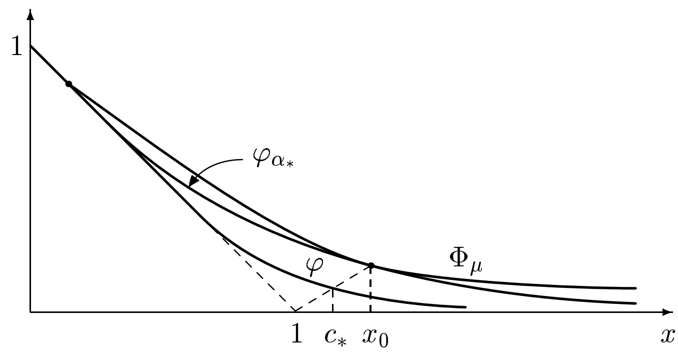

Figure 5. The form of c_*

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