ON TWO APPROACHES TO COHERENT RISK CONTRIBUTION

Alexander Cherny,[*1] ***Dmitri Orlov***[**]

Department of Probability Theory
Faculty of Mechanics and Mathematics
Moscow State University
119992 Moscow Russia
E-mail: `alexander.cherny@gmail.com`

[**]*Department of Probability Theory*
Faculty of Mechanics and Mathematics
Moscow State University
119992 Moscow Russia
E-mail: `orloff.dmitri@gmail.com`

Abstract. We compare between two approaches to coherent risk contribution: the *directional risk contribution* is defined as

$$\rho^d(X;Y) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[\rho(Y+\varepsilon X) - \rho(Y)],$$

where ρ is a coherent risk measure; the *linear risk contribution* $\rho^l(X;Y)$ is defined through a set of axioms, one of which is the linearity in X. The linear risk contribution exists and is unique for any ρ from the class *Weighted V@R*. We provide the representation for both risk contributions in the general setting as well as in some examples, including the *MINV@R* risk measure defined as

$$\mathrm{MINV@R}_N(X) = -\mathsf{E}\{X_1,\dots,X_N\},$$

where $X_1,\dots,X_N$ are independent copies of X.

Key words: Conditional V@R, coherent risk measure, directional risk contribution, linear risk contribution, minimal extreme measure, MINV@R, Weighted V@R.

1 Introduction

1. Risk contribution. Let X be a random variable meaning the cash flow produced over the unit time period by some portfolio and Y be the cash flow of another portfolio. A basic problem of risk measurement is to determine the risk contribution of X to Y. A number of recent investigations were devoted to this problem and to closely related problem of capital allocation in the framework of coherent risk measures introduced by Artzner et al. [3], [4]. Let us mention the papers by Artzner et al. [5], Cherny [7; Sect. 2.5],

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Delbaen [12; Sect. 9], Denault [13], Fischer [16], Kalkbrenner [18], Kalkbrenner et al. [19], Overbeck [21], Schmock [23; Sect. 3.9], and Tasche [24], [25; Sect. 4, 5]. However, different papers use different approaches to risk contributions and the definitions from different papers are not equivalent.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and ρ be a coherent risk measure defined as $\rho(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}X$, where $\mathcal{D}$ is a set of probability measures absolutely continuous with respect to P. One of the simplest ways of defining the risk contribution $\rho(X;Y)$ is to set

$$\rho(X;Y) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[\rho(Y+\varepsilon X) - \rho(Y)]. \tag{1.1}$$

This approach was taken in [7] and is very close to the approach of Denault [13], Fischer [16], and Tasche [24], [25], but there is an essential difference: we are considering in (1.1) the one-sided derivative ("$\varepsilon \downarrow 0$" means that $\varepsilon \to 0$ and $\varepsilon > 0$), while these authors consider the two-sided derivative. The two-sided derivative might not exist in natural situations (see Remark (ii) in Subsection 2.2). In contrast, the one-sided derivative always exists due to the convexity of the function $\varepsilon \mapsto \rho(Y+\varepsilon X)$. The quantity (1.1) is reasonable economically because it approximates the increase of the risk after we pass on from the portfolio Y to the portfolio $X+Y$, i.e. (1.1) can be informally interpreted as follows: if X is small as compared to Y, then

$$\rho(X+Y) \approx \rho(Y) + \rho(X;Y). \tag{1.2}$$

We call the risk contribution defined through (1.1) the *directional risk contribution* and denote it by $\rho^d(X;Y)$. It can be shown (see Theorem 2.1) that (under some minor technical conditions)

$$\rho^d(X;Y) = -\inf_{\mathsf{Q}\in\mathcal{X}(Y)} \mathsf{E}_{\mathsf{Q}}X, \tag{1.3}$$

where $\mathcal{X}(Y) = \operatorname{argmin}_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}Y$ is the set of elements of $\mathcal{D}$, at which the minimum of expectations $\mathsf{E}_{\mathsf{Q}}Y$ over $\mathcal{D}$ is attained (we call it the set of *extreme measures* for Y).

It is often the case that $\mathcal{X}(Y)$ is a singleton. For example, this is true if ρ is *Conditional V@R* (*CV@R*)[2] and Y has continuous distribution, i.e. $\mathsf{P}(X=x)=0$ for any x. Then $\rho^d(X;Y)$ is linear in X. However, in some natural situations $\mathcal{X}(Y)$ is not a singleton. For example, this might happen if ρ is CV@R and Y has atoms (see Subsection 3.2), which is typical for credit portfolios, where atoms of Y correspond to defaults. Then $\rho^d(X;Y)$ is no longer linear in X. On the other hand, the linearity is a very desirable property, especially in view of the capital allocation considerations. This is the reason why the CV@R risk contribution defined in Kalkbrenner [18], Kalkbrenner et al. [19], and Schmock [23] differ from (1.1); to be more precise, the two definitions differ only in the case when Y has atoms.

In this paper we provide an axiomatic definition of what we call the *linear risk contribution* $\rho^l(X;Y)$; see Definition 2.2. Some of our axioms coincide with those in Kalkbrenner [18], but our axiom system is essentially different than the one in [18]. Namely, Kalkbrenner considers two systems: one with a certain continuity assumption and the other is without this assumption. The first system determines the risk contribution uniquely,

[2] Recall that CV@R is defined as $\text{CV@R}_\lambda(X) = -\inf_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E}_{\mathsf{Q}}X$, where $\mathcal{D}_\lambda = \{\mathsf{Q} : d\mathsf{Q}/d\mathsf{P} \le \lambda^{-1}\}$ and $\lambda \in (0,1]$ is a fixed number. It is also known as the *Average V@R*, *Tail V@R*, and *Expected Shortfall*. For X with a continuous distribution, $\text{CV@R}_\lambda(X) = -\mathsf{E}[X \,|\, X \le q_\lambda(X)]$, where q_λ is a λ-quantile. For more information on this class, we refer to Acerbi and Tasche [2], Föllmer and Schied [17; Sect. 4.4], Rockafellar and Uryasev [22], and Tasche [25].

but it does not exist for all the pairs X, Y; for the second axiom system, the risk contribution exists in natural cases but is not unique (see Remark (ii) in Subsection 2.2). In contrast, as stated by Theorem 2.3, for our set of axioms, the risk contribution exists and is unique for any ρ from the class *Weighted V@R* (*WV@R*).[3] This class is very wide and, in our opinion, is sufficient for any practical application of coherent risks. As shown by Theorem 2.3,

$$\mathrm{WV@R}^l_\mu(X;Y) = -\mathsf{E}_{\mathsf{Q}_*(Y)}X, \tag{1.4}$$

where $\mathsf{Q}_*(Y)$ is a particular measure from the set $\mathcal{X}(Y)$. This measure was introduced in [8] under the name *minimal extreme measure* and was characterized as the unique element of $\mathcal{X}(Y)$ whose density $d\mathsf{Q}/d\mathsf{P}$ is minimal with respect to the second-order stochastic dominance. It is given explicitly by $d\mathsf{Q}_*(Y)/d\mathsf{P} = \varphi_\mu(Y)$, where

$$\varphi_\mu(y) = \int_{(D_Y(y-),D_Y(y))} \frac{1-\lambda^{-1}D_Y(y-)}{D_Y(y)-D_Y(y-)}\,\mu(d\lambda) + \int_{[D_Y(y),1]} \lambda^{-1}\mu(d\lambda), \quad y \in \mathbb{R}$$

and D_Y denotes the distribution function of Y.

CV@R is a subclass of WV@R and, for CV@R, our linear risk contribution coincides with the risk contribution considered in [18], [19], and [23] (see Corollary 3.2). However, in those papers $\mathrm{CV@R}^l$ appears as *some* functional satisfying certain axioms (in fact, any functional of the form $\rho(X;Y) = -\mathsf{E}_{\mathsf{Q}(Y)}X$ with $\mathsf{Q}(Y) \in \mathcal{X}(Y)$ satisfies those axioms as well), while our axioms characterize $\mathrm{CV@R}^l$ completely. Another difference is that WV@R is not considered in the above mentioned papers.

It is seen from (1.3) and (1.4) that

$$\mathrm{WV@R}^d_\mu(X;Y) \ge \mathrm{WV@R}^l_\mu(X;Y).$$

For Y with a continuous distribution, the two risk contributions are equal (see Corollary 2.4). However, if Y has atoms, then the above inequality might be strict (see Example 2.5). Random variables with atoms arise naturally in credit risk models (see, for example, [23]). Thus, the two risk contributions are in essence different.

2. Examples. CV@R is a basic example of coherent risks. Its advantage is that it is simple. As compared to V@R, it measures not only the probability of loss but its severity as well. However, its disadvantage is that it depends only on the tail of the distribution, i.e. it is a 0–1 risk measure.

WV@R is a much more flexible risk measure. If $\operatorname{supp}\mu = [0,1]$, where "supp" denotes the support, then $\mathrm{WV@R}_\mu$ depends on the whole distribution of X and penalizes losses in a "smooth" manner rather than 0–1 manner. This is an economical advantage of WV@R over CV@R. It has also a mathematical advantage: if $\operatorname{supp}\mu = [0,1]$, then $\mathrm{WV@R}_\mu(X+Y) < \mathrm{WV@R}_\mu(X) + \mathrm{WV@R}_\mu(Y)$ provided that X, Y are not comonotone[4] (see [8; Th. 5.1]). This leads to the uniqueness of a solution of certain optimization problems based on WV@R. However, the weak point of WV@R is that this class is too wide and the basic problem that arises immediately is: what measure μ to choose?

[3] Recall that WV@R is defined as $\mathrm{WV@R}_\mu(X) = \int_0^1 \mathrm{CV@R}_\lambda(X)\mu(d\lambda)$, where μ is a probability measure on $(0,1]$. It is also known as the *spectral risk measure*. For more information on WV@R, we refer to Acerbi [1], Cherny [8], Dowd [14], Föllmer and Schied [17; Sect. 4.6].

[4] Recall that X and Y are called *comonotone* if there exists a random variable Z and increasing functions f, g such that $X = f(Z)$, $Y = g(Z)$. For example, if X, Y have a joint density, then they are not comonotone.

One way to do that might be to derive the measure μ by comparing the physical measure (estimated from data) with the risk-neutral one (obtained from option prices); see Cherny and Madan [11]. The other way could be to introduce some nice subclasses. One such subclass was introduced by Cherny and Madan [9], [10] under the name *MINV@R*. It is defined as WV@R with $\mu(dx) = (N(N-1))^{-1}x(1-x)^{N-2}dx$ and has a very simple representation:

$$\text{MINV@R}_N(X) = -\mathsf{E}\{X_1, \dots, X_N\}, \tag{1.5}$$

where $X_1, \dots, X_N$ are independent copies of X.[5] Thus, MINV@R has the advantage over CV@R in being a smoother risk measure and retains the advantage of having a simple representation.

Theorem 3.1 provides a representation of WV@R contributions in the case of a finite Ω, which yields a numerical algorithm for the estimation of these risk contributions (its "projection" on CV@R is Corollary 3.3). The result of Theorem 3.4 provides both a numerical procedure and an elegant representation of MINV@R contributions. This theorem states that

$$\text{MINV@R}_N^d(X;Y) = -\mathsf{E}\min\{X_i : i \in \operatorname{argmin}_n Y_n\},$$

$$\text{MINV@R}_N^l(X;Y) = -\mathsf{E}\frac{\sum_{i\in\operatorname{argmin}_n Y_n} X_i}{|\operatorname{argmin}_n Y_n|},$$

where $(X_1, Y_1), \dots, (X_N, Y_N)$ are independent copies of (X, Y) and $|A|$ denotes the number of elements of A. In particular, if Y has no atoms, then $\operatorname{argmin}_n Y_n$ is a.s. a singleton, and we arrive at the representation obtained (under this assumption) in Cherny and Madan [9; Sect. 5]:

$$\text{MINV@R}_N^d(X;Y) = \text{MINV@R}_N^l(X;Y) = -\mathsf{E}X_{\operatorname{argmin}_n Y_n}.$$

3. Structure of the paper. In Section 2, we consider the directional risk contribution for general coherent risks and the linear risk contribution for WV@R. Section 3 provides the formulas for calculating risk contributions for WV@R in the discrete case, for CV@R, and for MINV@R. Section 4 concludes.

2 Risk Contribution

2.1 Directional Risk Contribution

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and ρ be a coherent risk measure defined by $\rho(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$, where $\mathcal{D}$ is a set of measures absolutely continuous with respect to P. We will assume that the set of Radon-Nikodym derivatives $\{d\mathsf{Q}/d\mathsf{P} : \mathsf{Q} \in \mathcal{D}\}$ is convex, L^1-closed, and uniformly integrable. This assumption is very mild; for example, it is automatically satisfied for the case when ρ is WV@R, as follows from [8; Th. 4.6].

Traditionally, coherent risks are defined on bounded random variables. However, this is insufficient for financial applications as most distributions (e.g. the Gaussian one) are unbounded. We will consider ρ on the space

$$L^1(\mathcal{D}) = \Big\{X \in L^0 : \lim_{n\to\infty} \sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q}|X|I(|X| > n) = 0\Big\}.$$

[5]The proof is actually very simple. One of equivalent representations of WV@R is: $\text{WV@R}_\mu(X) = -\mathsf{E}\widetilde{X}$, where $\widetilde{X}$ is a random variable with the distribution function $D_{\widetilde{X}} = \Psi_\mu \circ D_X$ and $\Psi_\mu(x) = \int_0^x \int_y^1 z^{-1}\mu(dz)dy$. For $\mu(dx) = (N(N-1))^{-1}x(1-x)^{N-2}dx$, we have $\Psi_\mu(x) = 1-(1-x)^N$, so that $\widetilde{X} \stackrel{\text{Law}}{=} \min\{X_1, \dots, X_N\}$.

It is easy to check that it is a linear space. For WV@R, we have a simpler representation (see [7; Prop. 2.6]):

$$L^1(\mathcal{D}) = \Big\{X \in L^0 : \sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q}|X| < \infty\Big\}.$$

If the weighting measure μ furthermore satisfies $\int_0^1 \lambda^{-1}\mu(d\lambda) < \infty$, as is the case for MINV@R, then $L^1(\mathcal{D}) = L^1$, as seen from [8; Th. 4.6].

Let $X, Y \in L^1(\mathcal{D})$. As the function $\varepsilon \mapsto \rho(Y+\varepsilon X)$ is finite and convex, there exists the directional risk contribution $\rho^d(X;Y)$ defined by (1.1). Furthermore, the set of extreme measures $\mathcal{X}(Y) = \operatorname{argmin}_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} Y$ is non-empty (see [7; Prop. 2.9]).

The next proposition provides the form of $\rho^d(X;Y)$. It is borrowed from [8]. As the proof is rather short, we repeat it here. The key idea is the introduction of the notion of a generator.

Theorem 2.1. *For $X, Y \in L^1(\mathcal{D})$, we have*

$$\rho^d(X;Y) = -\inf_{\mathsf{Q}\in\mathcal{X}(Y)} \mathsf{E}_\mathsf{Q} X.$$

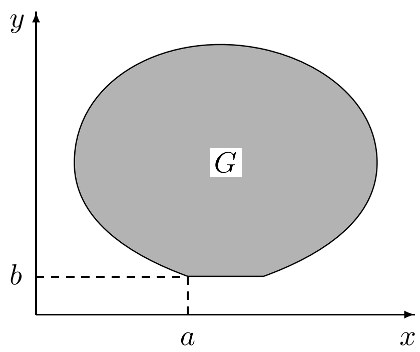

Figure 1

Proof. (The proof is illustrated by Figure 1.) The key step is to introduce the set

$$G = \operatorname{cl}\{\mathsf{E}_\mathsf{Q}(X,Y) : \mathsf{Q} \in \mathcal{D}\},$$

where "cl" denotes the closure. Obviously, G is a convex compact in $\mathbb{R}^2$. It is called the *generator* of X, Y, according to the terminology of [8]. The role of G is seen from the line

$$\begin{aligned}\rho(\alpha X + \beta Y) &= -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q}(\alpha X + \beta Y)\\ &= -\inf_{\mathsf{Q}\in\mathcal{D}} \langle(\alpha,\beta), \mathsf{E}_\mathsf{Q}(X,Y)\rangle\\ &= -\min_{x\in G}\langle(\alpha,\beta), x\rangle, \quad \alpha,\beta\in\mathbb{R}.\end{aligned}$$

Set $b = \min\{y : (x,y) \in G\}$, $a = \min\{x : (x,b) \in G\}$. For $\varepsilon > 0$, the minimum $\min_{(x,y)\in G}\langle(\varepsilon,1),(x,y)\rangle$ is attained at a point $(a(\varepsilon), b(\varepsilon))$. We obviously have $a(\varepsilon) \le a$, $b(\varepsilon) \ge b$, and $(a(\varepsilon), b(\varepsilon)) \xrightarrow[\varepsilon\downarrow 0]{} (a,b)$. Furthermore, $\varepsilon a(\varepsilon) + b(\varepsilon) \le \varepsilon a + b$, which implies that $0 \le b(\varepsilon) - b \le \varepsilon(a - a(\varepsilon))$. As a result,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[\rho(Y+\varepsilon X) - \rho(Y)] &= -\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}[\varepsilon a(\varepsilon) + b(\varepsilon) - b]\\ &= -a - \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}[b(\varepsilon) - b]\\ &= -a = -\inf_{\mathsf{Q}\in\mathcal{X}(Y)} \mathsf{E}_\mathsf{Q} X.\end{aligned}$$

2.2 Linear Risk Contribution

Let ρ be a coherent risk measure. In this subsection, we restrict attention to bounded random variables. This is a typical restriction imposed to obtain a representation result.

Definition 2.2. A *linear risk contribution* is a functional $\rho^l(X;Y)$ defined on $L^\infty \times L^\infty$ and satisfying the axioms:

(i) (Linearity) $\rho^l(a_1X_1 + a_2X_2; Y) = a_1\rho^l(X_1;Y) + a_2\rho^l(X_2;Y)$ for $a_1, a_2 \in \mathbb{R}$;
(ii) (Diversification) $\rho^l(X;Y) \le \rho(X)$;
(iii) (Consistency) $\rho^l(X;X) = \rho(X)$;
(iv) (Law invariance) $\rho^l(X;Y)$ depends only on the joint law of (X,Y);
(v) (Continuity in X) If $|X_n| \le 1$ and $X_n \xrightarrow{\mathsf{P}} X$, then $\rho^l(X_n;Y) \to \rho^l(X;Y)$.

Theorem 2.3 shows that the linear risk contribution exists and is unique at least for the case when ρ is WV@R. Recall that WV@R is defined as

$$\text{WV@R}_\mu(X) = \int_0^1 \text{CV@R}_\lambda(X)\mu(d\lambda), \quad X \in L^\infty,$$

where μ is a probability measure on $(0,1]$,

$$\text{CV@R}_\lambda(X) = -\inf_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X, \quad X \in L^\infty, \tag{2.1}$$

and $\mathcal{D}_\lambda = \{\mathsf{Q} : d\mathsf{Q}/d\mathsf{P} \le \lambda \le \lambda^{-1}\}$. For more information on this risk measure, we refer to [2], [8], and [17; Sect. 4.6].

WV@R$_\mu$ is a coherent risk measure, so that there exists a set $\mathcal{D}$ of probability measures absolutely continuous with respect to P such that $\text{WV@R}_\mu(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$, $X \in L^\infty$. We will denote by $\mathcal{D}_\mu$ the largest set $\mathcal{D}$, for which this representation is true. Measures from $\mathcal{D}_\mu$ (denoted below by Q) will be identified with their Radon-Nikodym derivatives with respect to P (denoted below by Z). For various representations of $\mathcal{D}_\mu$, see [6] and [8; Sect. 4]. In particular, it is seen from those representations that $\mathcal{D}_\mu$ is convex, L^1-closed, and uniformly integrable. As mentioned above, this guarantees that, for any $Y \in L^\infty$, the set $\mathcal{X}_\mu(Y) = \operatorname{argmin}_{\mathsf{Q}\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{Q} Y$ is non-empty. Furthermore, as shown in [8; Th. 6.3], there exists a particular representative Z_Y^* of $\mathcal{X}_\mu(Y)$ that is the smallest in the second-order stochastic dominance, i.e. $\mathsf{E}f(Z_Y^*) \le \mathsf{E}f(Z)$ for any convex function f of linear growth and any $Z \in \mathcal{X}_\mu(Y)$. The element Z_Y^* is unique and is given by $Z_Y^* = \varphi_\mu(Y)$, where

$$\varphi_\mu(y) = \int_{(D_Y(y-),D_Y(y))} \frac{1 - \lambda^{-1}D_Y(y-)}{D_Y(y) - D_Y(y-)}\mu(d\lambda) + \int_{[D_Y(y),1]} \lambda^{-1}\mu(d\lambda), \quad y \in \mathbb{R}.$$

We will denote the corresponding measure by $\mathsf{Q}_*(Y)$.

Theorem 2.3. *Suppose that the probability space is atomless.*[6] *For WV@R, the linear risk contribution exists, is unique, and is given by*

$$\text{WV@R}_\mu^l(X;Y) = -\mathsf{E}_{\mathsf{Q}_*(Y)} X = -\mathsf{E}[X\varphi_\mu(Y)]. \tag{2.2}$$

[6] Recall that this means that, for any $A \in \mathcal{F}$ with $\mathsf{P}(A) > 0$, there exist $A' \subseteq A$ such that $\mathsf{P}(A') > 0$ and $\mathsf{P}(A \setminus A') > 0$.

Proof. Obviously, (2.2) defines a linear risk contribution. Let us prove the uniqueness part. Let $\text{WV@R}^l_\mu(X;Y)$ be a linear risk contribution. Fix $Y \in L^\infty$.

For $r \in \mathbb{R}$, consider the set $C_r = \{X \in L^\infty : \text{WV@R}^l_\mu(X;Y) \le r\}$. Due to (v), for any n, the set $C_{r,n} = C_r \cap \{\|X\|_\infty \le n\}$ is closed in probability. Hence, it is L^1-closed (considered as a subset of L^1). Due to (i), it is convex, so by the Hahn-Banach theorem it is $\sigma(L^1, L^\infty)$-closed. As $L^\infty \subset L^1$, $C_{r,n}$ is $\sigma(L^\infty, L^1)$-closed. By the Krein-Smulian theorem (see [15; Th. V.5.7]) C_r is $\sigma(L^\infty, L^1)$-closed. This means that $\text{WV@R}^l_\mu(\,\cdot\,;Y)$ is $\sigma(L^\infty, L^1)$-continuous. As this functional is linear, there exists $Z \in L^1$ such that $\text{WV@R}^l_\mu(X;Y) = -\mathsf{E}ZX$ for any $X \in L^\infty$.

Suppose that $Z \notin \mathcal{D}_\mu$. As $\mathcal{D}\mu$ is convex and L^1-closed, then, by the Hahn-Banach theorem, there exists $X \in L^\infty$ such that $\mathsf{E}ZX < \inf_{Z \in \mathcal{D}_\mu} \mathsf{E}ZX$. But this contradicts (ii). Consequently, $Z \in \mathcal{D}_\mu$.

Taking (iii) into consideration, we see that $Z \in \mathcal{X}_\mu(Y)$.

According to [8; Prop. 6.2], there exists a $\mathcal{B}((0,1]) \times \mathcal{F}$-measurable function $Z(\lambda, \omega)$ such that $Z = \int_0^1 Z_\lambda \mu(d\lambda)$ (here $Z_\lambda(\omega) = Z(\lambda, \omega)$) and

$$\begin{cases} Z_\lambda = \lambda^{-1} & \text{a.e. on } \{Y < q_\lambda(Y)\}, \\ 0 \le Z_\lambda \le \lambda^{-1} & \text{a.e. on } \{Y = q_\lambda(Y)\}, \\ Z_\lambda = 0 & \text{a.e. on } \{Y > q_\lambda(Y)\}, \end{cases} \tag{2.3}$$

where q_λ denotes a λ-quantile. For each $\lambda \in (0,1]$, consider the random variable

$$Z^*_\lambda = \begin{cases} \lambda^{-1} & \text{on } \{Y < q_\lambda(Y)\}, \\ c_\lambda & \text{on } \{Y = q_\lambda(Y)\}, \\ 0 & \text{on } \{Y > q_\lambda(Y)\}, \end{cases} \tag{2.4}$$

where c_λ is chosen in such a way that $\mathsf{E}Z^*_\lambda = 1$. Define $Z^* = \int_0^1 Z^*_\lambda \mu(d\lambda)$. We can represent Ω as a disjoint union $A \cup (\bigcup_{n=1}^\infty A_n)$ so that Y has a continuous distribution on A and Y is constant on each A_n. Note that, for any n, $Z_\lambda = Z^*_\lambda$ a.e. on A. Hence, $Z = Z^*$ a.e. on A.

Fix n and suppose that Z is not constant on A_n. As the probability space is atomless, there exist sets $B, B' \subseteq A_n$ such that $\mathsf{P}(B) = \mathsf{P}(B')$ and $\mathsf{E}I(B)Z \ne \mathsf{E}I(B')Z$. But then, for $X = I_B$ and $X' = I_{B'}$, we have $(X,Y) \overset{\text{Law}}{=} (X', Y)$, while $\mathsf{E}ZX \ne \mathsf{E}ZX'$, which contradicts (iv). As a result, Z is a.e. constant on each A_n.

Note that each Z^*_λ is Y-measurable. Hence, Z^* is Y-measurable. As $Z = Z^*$ a.e. on A and Z is constant on each A_n, we see that Z is Y-measurable. Obviously, $\mathsf{E}(Z_\lambda | Y) = Z^*_\lambda$. As a result,

$$Z = \mathsf{E}(Z|Y) = \int_0^1 \mathsf{E}(Z_\lambda|Y)\mu(d\lambda) = \int_0^1 Z^*_\lambda \mu(d\lambda) = Z^*.$$

It remains to note that $Z^*_\lambda = \varphi_\lambda(Y)$, where

$$\varphi_\lambda(y) = \frac{1 - \lambda^{-1} D_Y(y-)}{D_Y(y) - D_Y(y-)} I(D_Y(y-) < \lambda < D_Y(y)) + \lambda^{-1} I(\lambda \ge D_Y(y)).$$

Clearly, $\int_0^1 \varphi_\lambda(y)\mu(d\lambda) = \varphi_\mu(y)$. Thus,

$$\text{WV@R}^l_\mu(X;Y) = -\mathsf{E}ZX = -\mathsf{E}Z^*X = \mathsf{E}[X\varphi_\mu(Y)].$$

Corollary 2.4. *If Y has a continuous distribution, then $\mathcal{X}(Y)$ is a singleton and*

$$\mathrm{WV@R}^d_\mu(X;Y) = \mathrm{WV@R}^l_\mu(X;Y) = -\mathsf{E}\biggl[X\int_{[D_Y(Y),1]}\lambda^{-1}\mu(d\lambda)\biggr].$$

With no continuity assumption on the law of Y, the above statement is wrong, as shown by the next example.

Example 2.5. Let $\mu = \delta_\lambda$, i.e. ρ is $\mathrm{CV@R}_\lambda$. Let $Y \in L^\infty$ be such that $\mathsf{P}(Y < q_\lambda(Y)) < \lambda$ and $\mathsf{P}(Y \le q_\lambda(Y)) > \lambda$, i.e. Y has an atom at its λ-quantile. Then $\mathcal{X}_\mu(Y)$ consists of the densities satisfying (2.3), while $\mathsf{Q}_*(Y)$ has the density given by (2.4) (see [8]). Thus, there exists $X \in L^\infty$ such that

$$\mathrm{CV@R}^d_\lambda(X;Y) = -\inf_{\mathsf{Q}\in\mathcal{X}_\mu(Y)}\mathsf{E}_\mathsf{Q}X > -\mathsf{E}_{\mathsf{Q}_*(Y)} = \mathrm{CV@R}^l_\lambda(X;Y).$$

For example, if $Y = 0$, then $\mathcal{X}_\mu(Y) = \mathcal{D}$ and $\mathsf{Q}_*(Y) = \mathsf{P}$, so that $\mathrm{CV@R}^d_\lambda(X;Y) = \mathrm{CV@R}_\lambda(X)$, while $\mathrm{CV@R}^l_\lambda(X;Y) = -\mathsf{E}X$.

Remarks. (i) Representation (2.2) allows us to extend $\mathrm{WV@R}^l_\mu$ to $X, Y \in L^1(\mathcal{D}_\mu)$. This is similar to the situation with coherent risk measures: the representation theorem is proved on L^∞, and then, using this representation, one can extend a risk measure to a wider class of random variables (as was done at the beginning of Subsection 2.1).

(ii) Conditions (i)–(iii) of Definition 2.2 are the same as those in Kalkbrenner [18]. He also introduces a continuity in Y assumption requiring that

$$\lim_{\varepsilon\to 0}\rho(X;Y+\varepsilon X) = \rho(X;Y).$$

As proved in [18; Th. 3.1], if $\rho(X;Y)$ satisfies (i)–(iii) and the above continuity assumption, then

$$\rho(X;Y) = \lim_{\varepsilon\to 0}\varepsilon^{-1}[\rho(Y+\varepsilon X) - \rho(Y)] \tag{2.5}$$

(note that here ε is not assumed to be positive). However, $\rho(X;Y)$ determined this way does not always exist. Indeed, in the framework of Example 2.5,

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}[\rho(Y+\varepsilon X) - \rho(Y)] = -\inf_{\mathsf{Q}\in\mathcal{X}_\mu(Y)}\mathsf{E}_\mathsf{Q}X,$$
$$\lim_{\varepsilon\downarrow 0}(-\varepsilon)^{-1}[\rho(Y-\varepsilon X) - \rho(Y)] = -\sup_{\mathsf{Q}\in\mathcal{X}_\mu(Y)}\mathsf{E}_\mathsf{Q}X,$$

so that limit (2.5) does not exist.

On the other hand, conditions (i)–(iii) without continuity in Y do not identify the risk contribution uniquely. Again in the framework of Example 2.5 if we choose for each $Y \in L^\infty$ a representative $\mathsf{Q}(Y)$ from $\mathcal{X}_\mu(Y)$, then the functional $\rho(X;Y) = -\mathsf{E}_{\mathsf{Q}(Y)}X$ satisfies (i)–(iii).

(iii) Unlike [18], we do not impose in our axiomatics any assumption of the continuity of $\rho(X;Y)$ in Y. Nevertheless, a certain continuity in Y does hold for $\mathrm{WV@R}^l_\mu(X;Y)$. Namely, if $X, Y \in L^\infty$ and (Z_n) is a family of bounded random variables such that each Z_n is independent of (X,Y) and $Z_n \xrightarrow{\mathsf{P}} 0$, then $\mathrm{WV@R}^l_\mu(X;Y+Z_n) \xrightarrow[n\to\infty]{} \mathrm{WV@R}^l_\mu(X;Y)$ (see [8; Th. 6.5]).

(iv) It is clear from conditions (iii) and (iv) of Definition 2.2 that ρ^l can exist only for a law invariant risk measure ρ, i.e. $\rho(X)$ should depend only on the law of X. The class of law invariant risk measures is wider than WV@R: the result of Kusuoka [20] states that (on an atomless probability space) a coherent risk measure ρ is law invariant if and only if it can be represented as $\inf_{\mu\in\mathfrak{M}} \mathrm{WV@R}_\mu(X)$ with a set $\mathfrak{M}$ of probability measures on $(0,1]$. However, for a general law invariant risk measure, both the existence and the uniqueness of ρ^l might be violated.

For example, let $\rho(X) = -\inf_{\mathsf{Q}\in\mathcal{D}_0} \mathsf{E}_\mathsf{Q} X$, where $\mathcal{D}_0$ is the set of all probability measures absolutely continuous with respect to P. Then $\rho(X) = -\operatorname{essinf}_\omega X(\omega)$ and if $\mathsf{P}(Y = \operatorname{essinf}_\omega Y(\omega)) = 0$, then $\operatorname{argmin}_{\mathsf{Q}\in\mathcal{D}_0} \mathsf{E}_\mathsf{Q} Y = \emptyset$. On the other hand, the proof of Theorem 2.3 shows that $\rho^l(X;Y)$ should be represented as $-\mathsf{E}_{\mathsf{Q}(Y)}X$ with $\mathsf{Q}(y) \in \operatorname{argmin}_{\mathsf{Q}\in\mathcal{D}_0} \mathsf{E}_\mathsf{Q} Y$. So, in this case ρ^l does not exist.

To construct an example with the non-uniqueness of ρ^l, consider $\rho(X) = \mathrm{WV@R}_{\mu_1}(X) \vee \mathrm{WV@R}_{\mu_2}(X)$. Denote by φ_1 and φ_2 the function φ_μ corresponding to μ_1 and μ_2, respectively. If we take

$$Z_Y = \begin{cases} \varphi_1(Y) & \text{if } \mathrm{WV@R}_{\mu_1}(Y) > \mathrm{WV@R}_{\mu_2}(Y), \\ \varphi_2(Y) & \text{if } \mathrm{WV@R}_{\mu_1}(Y) \le \mathrm{WV@R}_{\mu_2}(Y), \end{cases}$$

$$Z'_Y = \begin{cases} \varphi_1(Y) & \text{if } \mathrm{WV@R}_{\mu_1}(Y) \ge \mathrm{WV@R}_{\mu_2}(Y), \\ \varphi_2(Y) & \text{if } \mathrm{WV@R}_{\mu_1}(Y) < \mathrm{WV@R}_{\mu_2}(Y), \end{cases}$$

then both functionals $-\mathsf{E}Z_Y X$ and $-\mathsf{E}Z'_Y X$ are linear risk contributions.

3 Examples

3.1 Weighted V@R in Discrete Case

Let $\Omega = \{1,\dots,M\}$ endowed with the uniform measure. The typical situation is that we have M empirical realizations $(x_1,y_1),\dots,(x_M,y_M)$ of (X,Y) and replace the true distribution by the empirical one, i.e. we consider the random variables defined as $X(m) = x_m$, $Y(m) = y_m$, $m\in\Omega$. Let $n(m)$ be a permutation of $\{1,\dots,M\}$ such that the sequence $Y(n(m))$ is increasing and $X(n(m))$ is increasing over any interval of $\{1,\dots,M\}$, over which $Y(n(m))$ is constant. Let $0 = l_0 < l_1 < \dots < l_K = M$ be numbers such that $Y(n(m))$ is constant for $m = l_{k-1}+1,\dots,l_k$ and $Y(n(l_k)) < Y(n(l_k+1))$.

Theorem 3.1. *We have*

$$\mathrm{WV@R}^d_\mu(X;Y) = -\sum_{m=1}^M X(n(m))\Big[\Psi_\mu\Big(\frac{m}{M}\Big) - \Psi_\mu\Big(\frac{m-1}{M}\Big)\Big], \tag{3.1}$$

$$\mathrm{WV@R}^l_\mu(X;Y) = -\sum_{k=1}^K \sum_{m=l_{k-1}+1}^{l_k} \frac{X(n(m))}{l_k - l_{k-1}}\Big[\Psi_\mu\Big(\frac{l_k}{M}\Big) - \Psi_\mu\Big(\frac{l_{k-1}}{M}\Big)\Big], \tag{3.2}$$

where

$$\Psi_\mu(x) = \int_0^x \int_y^1 z^{-1}\mu(dz)dy, \quad x\in[0,1].$$

In particular, if $Y(1),\dots,Y(M)$ are different, then

$$\mathrm{WV@R}^d_\mu(X;Y) = \mathrm{WV@R}^l_\mu(X;Y) = -\sum_{m=1}^M X(n(m))\Big[\Psi_\mu\Big(\frac{m}{M}\Big) - \Psi_\mu\Big(\frac{m-1}{M}\Big)\Big],$$

where $n(m)$ is the unique permutation of $\{1,\dots,M\}$ such that $Y(n(m))$ is increasing.

Proof. One of equivalent representations of WV@R, which follows from [17; Th. 4.64], is as follows: $\mathrm{WV@R}_\mu(Z) = -\mathsf{E}\widetilde{Z}$, where $\widetilde{Z}$ has the distribution function $\Psi_\mu \circ D_Z$. It is seen from this representation that in our case we have for any random variable Z on Ω,

$$\mathrm{WV@R}_\mu(Z) = -\sum_{m=1}^{M} Z(i(m))\left[\Psi_\mu\left(\frac{m}{M}\right) - \Psi_\mu\left(\frac{m-1}{M}\right)\right], \tag{3.3}$$

where $i(m)$ is any permutation of $\{1,\dots,M\}$ such that $Z(i(m))$ is increasing.

It is obvious that, for the permutation $n(m)$, there exists $\delta > 0$ such that, for any $\varepsilon \in [0,\delta]$, the sequence $(Y(n(m)) + \varepsilon X(n(m)))$ is increasing. Then, for any $\varepsilon \in [0,\delta]$,

$$\mathrm{WV@R}_\mu(Y+\varepsilon X) = -\sum_{m=1}^{M}[Y(n(m)) + \varepsilon X(n(m))]\left[\Psi_\mu\left(\frac{m}{M}\right) - \Psi_\mu\left(\frac{m-1}{M}\right)\right],$$

from which (3.1) is obvious.

The functional $\rho(X;Y)$ defined by the right-hand side of (3.2) (it is meant that the permutation $n(m)$ depends on (X,Y)) satisfies conditions (i), (iv), and (v) of Definition 2.2. It follows from (3.3) that it also satisfies (iii). To prove (ii), consider the function

$$f : \{1,\dots,M\} \ni m \longmapsto \frac{1}{l_{k(m)} - l_{k(m)-1}}\left[\Psi_\mu\left(\frac{l_{k(m)}}{M}\right) - \Psi_\mu\left(\frac{l_{k(m)-1}}{M}\right)\right],$$

where $k(m)$ is such that $l_{k(m)-1} < m \le l_{k(m)}$. As Ψ_μ is concave, f is decreasing. A discrete version of the Hardy-Littlewood inequality (see [17; Th. A.24]) shows that

$$\sum_{m=1}^{M} X(n(m))f(m) \ge \sum_{m=1}^{M} X(i(m))f(m),$$

where $i(m)$ is a permutation of $\{1,\dots,M\}$ such that $X(i(m))$ is increasing. It is obvious that

$$\sum_{m=1}^{M} X(i(m))f(m) = \sum_{m=1}^{M} X(i(m))\left[\Psi_\mu\left(\frac{m}{M}\right) - \Psi_\mu\left(\frac{m-1}{M}\right)\right].$$

Recalling (3.3), we see that condition (ii) of Definition 2.2 is satisfied. Now, an application of Theorem 2.3 yields (3.2). □

3.2 Conditional V@R

CV@R is defined by (2.1). It is a particular case of WV@R with $\mu = \delta_\lambda$. It is easy to check that the set $\mathcal{D}_{\delta_\lambda}$ coincides with the set $\mathcal{D}_\lambda$ standing in (2.1). It is clear that $L^1(\mathcal{D}_\lambda) = L^1$. For $Y \in L^1$, the set $\mathcal{X}_\lambda(Y) = \mathrm{argmin}_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} Y$ consists of the densities Z such that $Z = \lambda^{-1}$ a.e. on $\{Y < q_\lambda(Y)\}$, $0 \le Z \le \lambda^{-1}$ a.e. on $\{Y = q_\lambda(Y)\}$, and $Z = 0$ a.e. on $\{Y > q_\lambda(Y)\}$ (see [8; Prop. 2.7]). The measure $\mathsf{Q}_*(Y)$ has the density

$$Z_\lambda^* = \lambda^{-1} I(Y < q_\lambda(Y)) + \frac{1 - \lambda^{-1}D_Y(q_\lambda(Y)-)}{D_Y(q_\lambda(Y)) - D_Y(q_\lambda(Y)-)}\, I(Y = q_\lambda(Y)).$$

For $X, Y \in L^1$, we define $\mathrm{CV@R}_\lambda(X;Y)$ by (2.2). Then Theorem 2.3 yields

Corollary 3.2. *For $X, Y \in L^1$, we have*

$$\mathrm{CV@R}^l_\lambda(X;Y) = -\lambda^{-1}\mathsf{E}[XI(Y < q_\lambda(Y))] - \frac{1-\lambda^{-1}D_Y(q_\lambda(Y)-)}{D_Y(q_\lambda(Y)) - D_Y(q_\lambda(Y)-)}\mathsf{E}[XI(Y = q_\lambda(Y))].$$

In particular, if Y has a continuous distribution, then

$$\mathrm{CV@R}^d_\lambda(X;Y) = \mathrm{CV@R}^l_\lambda(X;Y) = -\mathsf{E}[X \,|\, Y \le q_\lambda(Y)].$$

Let us now consider the setting of Subsection 3.1 and assume that $\lambda = N/M$. Find k such that $l_{k-1} < N \le l_k$. Then Theorem 3.1 yields

Corollary 3.3. *We have*

$$\mathrm{CV@R}^d_\lambda(X;Y) = -\frac{1}{N}\sum_{m=1}^{N} X(n(m)),$$
$$\mathrm{CV@R}^l_\lambda(X;Y) = -\frac{1}{N}\sum_{m=1}^{l_{k-1}} X(n(m)) - \frac{N-l_{k-1}}{N(l_k-l_{k-1})}\sum_{m=l_{k-1}+1}^{l_k} X(n(m)).$$

In particular, if $Y(1),\dots,Y(M)$ are different, then

$$\mathrm{CV@R}^d_\lambda(X;Y) = \mathrm{CV@R}^l_\lambda(X;Y) = -\frac{1}{N}\sum_{m=1}^{N} X(n(m)),$$

where $n(m)$ is the unique permutation of $\{1,\dots,M\}$ such that $Y(n(m))$ is increasing.

3.3 MINV@R

MINV@R is a particular case of WV@R with $\mu(dx) = (N(N-1))^{-1}x(1-x)^{N-2}dx$, where N is a fixed natural number. For this μ, we have $\int_0^1 \lambda^{-1}\mu(d\lambda) < \infty$, which means that any density from $\mathcal{D}_\mu$ is bounded; see [8; Th. 4.6]. It is also seen from the representation of $\mathcal{D}_\mu$ that $\mathsf{P} \in \mathcal{D}_\mu$. As a result, in this case, $L^1(\mathcal{D}_\mu) = L^1$.

For $X, Y \in L^1$, we define $\mathrm{MINV@R}^l_N(X;Y)$ by (2.2). The expectation $\mathsf{E}_{\mathsf{Q}_*(Y)}X$ exists since $d\mathsf{Q}_*(Y)/d\mathsf{P}$ is bounded.

Theorem 3.4. *For $X, Y \in L^1$, we have*

$$\mathrm{MINV@R}^d_N(X;Y) = -\mathsf{E}\min\{X_i : i \in \mathrm{argmin}_n Y_n\}, \tag{3.4}$$
$$\mathrm{MINV@R}^l_N(X;Y) = -\mathsf{E}\frac{\sum_{i\in\mathrm{argmin}_n Y_n} X_i}{|\,\mathrm{argmin}_n Y_n|}, \tag{3.5}$$

where $(X_1,Y_1),\dots,(X_N,Y_N)$ are independent copies of (X,Y) and $|A|$ denotes the number of elements of A. In particular, if Y has a continuous distribution, then

$$\mathrm{MINV@R}^d_N(X;Y) = \mathrm{MINV@R}^l_N(X;Y) = -\mathsf{E}X_{\mathrm{argmin}_n Y_n}.$$

Proof. For $X, Y \in L^1$, we have

$$\varepsilon^{-1}[\min\{Y_1+\varepsilon X_1,\dots,Y_N+\varepsilon X_N\} - \min\{Y_1,\dots,Y_N\}] \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} \min\{X_i : i \in \mathrm{argmin}_n Y_n\}$$

and

$$|\varepsilon^{-1}[\min\{Y_1+\varepsilon X_1,\dots,Y_N+\varepsilon X_N\}-\min\{Y_1,\dots,Y_N\}]|\le\sum_{n=1}^N|X_n|.$$

Recalling representation (1.5), we get (3.4).

Let us prove (3.5) first for bounded X, Y. Define a map $\rho(X;Y)$ on $L^\infty\times L^\infty$ by the right-hand side of (3.5). This map obviously satisfies conditions (ii), (iii), and (iv) of Definition 2.2. Let us check (i). Fix $X_1, X_2, Y\in L^\infty$ and let $(X_{1n}, X_{2n}, Y)_{n=1}^N$ be independent copies of (X_1, X_2, Y). Then $(a_1X_{1n}+a_2X_{2n}, Y_n)_{n=1}^N$ are independent copies of $(a_1X_1+a_2X_2, Y)$. Furthermore,

$$a_1\frac{\sum_{i\in\operatorname{argmin}_n Y_n}X_{1i}}{|\operatorname{argmin}_n Y_n|}+a_2\frac{\sum_{i\in\operatorname{argmin}_n Y_n}X_{2i}}{|\operatorname{argmin}_n Y_n|}=\frac{\sum_{i\in\operatorname{argmin}_n Y_n}(a_1X_{1i}+a_2X_{2i})}{|\operatorname{argmin}_n Y_n|},$$

so that (i) is satisfied.

To check (v), consider X, Y, and $X_k\xrightarrow{\mathsf{P}}X$ with $|X_k|\le 1$. Let $(X_n, Y_n, X_{1n}, X_{2n},\dots)_{n=1}^N$ be independent copies of the infinite-dimensional vector $(X, Y, X_1, X_2,\dots)$. Then

$$\left|\frac{\sum_{i\in\operatorname{argmin}_n Y_n}X_{ki}}{|\operatorname{argmin}_n Y_n|}-\frac{\sum_{i\in\operatorname{argmin}_n Y_n}X_{i}}{|\operatorname{argmin}_n Y_n|}\right|\le\sum_{n=1}^N|X_{kn}-X_n|. \tag{3.6}$$

As $\mathsf{E}\sum_n|X_{kn}-X_n|\xrightarrow[k\to\infty]{}0$, we get $\rho(X_k;Y)\xrightarrow[k\to\infty]{}\rho(X;Y)$. Thus, (v) is proved and Theorem 2.3 yields (3.5) for bounded X, Y.

Let now $X\in L^1, Y\in L^\infty$. Consider $X_k=(-k)\vee X\wedge k$. As $d\mathsf{Q}_*(Y)/d\mathsf{P}$ is bounded, we have $\text{MINV@R}^l_N(X_k;Y)\to\text{MINV@R}^l_N(X;Y)$. The estimate (3.6) shows the convergence of the right-hand sides of (3.5). Thus, we get (3.5) for $X\in L^1, Y\in L^\infty$.

Finally, for $X\in L^1, Y\in L^1$, consider $\widetilde{Y}=\arctan Y$. The density $d\mathsf{Q}_*(Y)/d\mathsf{P}$ is a function of $D_Y(Y)$, while $d\mathsf{Q}_*(\widetilde{Y})/d\mathsf{P}$ is the same function applied to $D_{\widetilde{Y}}(\widetilde{Y})$. It is clear that $D_{\widetilde{Y}}(\widetilde{Y})=D_Y(Y)$, so that $\text{MINV@R}^l_N(X;\widetilde{Y})=\text{MINV@R}^l_N(X;Y)$. It is obvious that the right-hand sides of (3.5) coincide for (X, Y) and $(X,\widetilde{Y})$. Thus, (3.5) is proved. □

The above theorem provides a way for the numerical estimation of MINV@R contributions. In order to apply MINV@R, we first have to choose N. One way to do that could be to take a benchmark distribution X and a benchmark risk measure ρ and to find N, for which $\text{MINV@R}_N(X)$ is closest to $\rho(X)$. If X has a density p_X, then the formula for $\text{MINV@R}_N(X)$ is very simple:

$$\begin{aligned}\text{MINV@R}_N(X)&=-\int_{\mathbb{R}}xd\Psi_\mu(D_X(x))\\&=-\int_{\mathbb{R}}xd(1-(1-D_X(x)))^N\\&=-\int_{\mathbb{R}}Nx(1-D_X(x))^{N-1}p_X(x)dx\end{aligned}$$

(the first equality follows from [17; Th. 4.64]). For example, if X is Gaussian and ρ is 5%-V@R, then $N=12$.

Then one should take time series for X, Y and divide it into blocks of length N. Within each block, one should find the places, at which the smallest realizations of Y stand. In order to estimate $\text{MINV@R}^d_N(X,Y)$, one should take the smallest realization of X over those places and then take the average over the blocks with the minus sign. In order to estimate $\mathrm{MINV@R}_N^l(X;Y)$, one should first average the realizations of X over the places with the smallest Y within each block and then take the average over the blocks with the minus sign.

4 Conclusion

We have considered two approaches to the coherent risk contribution. The directional risk contribution $\rho^d(X;Y)$ for a risk measure $\rho(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$ is defined as the directional derivative of the risk ρ at the point Y in the direction X. It exists under very mild assumptions and equals $-\inf_{\mathsf{Q}\in\mathcal{X}(Y)} \mathsf{E}_\mathsf{Q} Y$, where $\mathcal{X}(Y) = \operatorname{argmin}_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} Y$ is the set of extreme measures.

The linear risk contribution $\rho^l(X;Y)$ was defined through a set of axioms. For WV@R, which is a very wide class of coherent risks, $\rho^l(X;Y)$ exists, is unique, and is given by $-\mathsf{E}_{\mathsf{Q}_*(Y)} X$, where $\mathsf{Q}_*(Y)$ is a particular representative of $\mathcal{X}(Y)$. If Y has a continuous distribution, then $\rho^d(X;Y) = \rho^l(X;Y)$. However, if Y has atoms, which is typical for credit risk considerations, then the two risk contributions are different and we only have the inequality $\mathrm{WV@R}_\mu^d(X;Y) \geq \mathrm{WV@R}_\mu^l(X;Y)$.

A particularly simple representative of WV@R is MINV@R defined as the expectation of the minimum of N independent draws from the distribution of X. Its advantage over CV@R is that it is a smoother risk measure. It was shown that both the directional and the linear MINV@R contributions admit simple representations, which also provides an empirical estimation procedure.

An algorithm for the empirical estimation of WV@R contributions has also been provided.

Each of the two risk contributions have its own advantages: the property (1.2) for the directional risk contribution and the linearity property for the linear risk contribution. Maybe, the latter property is more important because it is desirable for the capital allocation considerations, where the sum of risk contributions of several subportfolios should be equal to the risk of the whole portfolio.

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