RISK-REWARD OPTIMIZATION
WITH DISCRETE-TIME COHERENT RISK

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Abstract. We solve the risk-reward optimization problem in the discrete-time setting, the reward being measured as the expected Profit and Loss and the risk being measured by a dynamic coherent risk measure.

Key words and phrases. Conditionally Gaussian model, dynamic coherent risk measure, dynamic Weighted V@R, risk-reward optimization.

1 Introduction

Overview. Let $S_0^1, \dots, S_0^d$ be real numbers meaning the initial prices of several assets and let $S_1^1, \dots, S_1^d$ be random variables meaning their terminal discounted prices. In 1952 Markowitz [25] introduced the mean-variance optimization problem

$$\begin{cases} \mathsf{E}X \longrightarrow \max, \\ X \in \mathcal{A}, \\ \mathsf{Var}\, X \le c, \end{cases} \tag{1.1}$$

where $\mathcal{A} = \big\{\sum_i h^i(S_1^i - S_0^i) : h^i \in \mathbb{R}\big\}$ is the set of P&Ls (Profits and Losses) that can be obtained by various trading strategies, c is a risk constraint, and Var denotes variance.[2]

Although widely used and highly analytically tractable, variance has some deficiencies: it penalizes high profits in exactly the same way as high losses. In particular, the quadratic risk of any position coincides with the risk of the opposite position, which is not

[1] I am thankful to two anonymous Referees for a very careful reading of the paper and a number of valuable comments and suggestions that improved its quality.

[2] The standard formulation of the problem is in terms of returns, but it is more convenient for us to deal with the equivalent formulation in terms of discounted P&Ls. Let us also remark that in the original paper [25] Markowitz considered a model with no short sales, i.e. with $h^i \ge 0$.

relevant unless the payoff is symmetric. In order to overcome this problem, Markowitz [26] proposed *semivariance* $\mathsf{Svar}\, X = \mathsf{E}((X - \mathsf{E}X)^-)^2$ as a measure of risk. Although wiser than variance, this risk measure is much less convenient analytically: if one considers problem (1.1) with Var replaced by Svar, then one can prove the existence of a solution (see the recent paper [23] by Jin, Markowitz, and Zhou), but no analytical solution is available.

Another way for measuring risk smartly has recently been proposed by Artzner, Delbaen, Eber, and Heath [3], [4]. They introduced the concept of a *coherent risk measure.* Semivariance is in fact a particular representative of this class (to be more precise, the functional $-\mathsf{E}X + \alpha\, \mathsf{Svar}\, X$ with $\alpha \in [0,1]$ is a coherent risk measure; see [18]). But semivariance is not the most convenient representative.

A basic coherent risk measure is *Tail V@R* (known also as the *Average V@R*, *Conditional V@R*, and *Expected Shortfall*) defined as[3]

$$\rho_\lambda(X) = -\mathsf{E}(X \,|\, X \le q_\lambda), \quad \lambda \in (0,1],$$

where q_λ denotes the λ-quantile of X (see [20; Sect. 4.4] for basic properties of ρ_λ). A more general class is *Weighted V@R* (known also as the *spectral risk measure*)

$$\rho_\mu(X) = \int_{(0,1]} \rho_\lambda(X)\mu(d\lambda), \quad \mu \text{ is a probability measure on } (0,1].$$

This class is convenient analytically (see [1], [10]). Let us also mention one more family of coherent risks — *MINV@R* introduced in [14]: if one chooses $\mu(dx) = cx(1-x)^{N-2}$, where $N \in \mathbb{N}$, then ρ_μ takes the form

$$\rho_N(X) = -\mathsf{E}\min\{X_1, \dots, X_N\},$$

where $X_1, \dots, X_N$ are independent copies of X. Let us remark that the above mentioned risk measure $-\mathsf{E}X + \alpha\, \mathsf{Svar}\, X$ does not belong to the Weighted V@R class, and, in our opinion, this is the reason why it is not convenient analytically.

Thus, it is natural to consider the analog of (1.1)

$$\begin{cases} \mathsf{E}X \longrightarrow \max, \\ X \in \mathcal{A}, \\ \rho(X) \le c, \end{cases} \tag{1.2}$$

in which ρ is a coherent risk. This problem was studied in a number of papers. Rockafellar and Uryasev [28] showed that, for the case of Tail V@R, this problem is equivalent to

[3] To be more precise, this representation is true only if X has a continuous distribution; otherwise, the formula for ρ_λ is slightly more complicated.

another one, which is much more convenient for numeric computations (see Acerbi and Simonetti [2] for a generalization of this method to Weighted V@R). A solution of (1.2) in geometric terms was proposed in [12].

Goal of the paper. In this paper we study the dynamic discrete-time analog of (1.2), i.e. the problem, in which

$$\mathcal{A} = \left\{ \sum_{n=1}^{N} \sum_{i=1}^{d} H_n^i (S_n^i - S_{n-1}^i) : H_n^i \text{ is } \mathcal{F}_{n-1}\text{-measurable} \right\}, \tag{1.3}$$

where S_n^i means the discounted price of the i-th asset at time n; S is adapted to a filtration $(\mathcal{F}_n)_{n=0,\dots,N}$.

If one takes ρ to be the same as above (Tail V@R, Weighted V@R, etc.), then the resulting dynamic problem appears to be inconvenient: it might be possible to prove the existence and the uniqueness of a solution (under some conditions), but there seems to be no analytic solution even for basic models and no efficient numeric procedure for finding a solution. The reason is that the corresponding problem is not dynamically consistent.

But along with static coherent risks, there exist dynamic coherent risks (their study is one of the major topics in today's research in the field of risk measures; see [5], [6], [8], [7], [9], [17], [19], [21], [22], [24], [27], [29], [31]). It turns out that if one plugs in (1.2)–(1.3) a dynamic coherent risk, then the problem becomes dynamically consistent and admits a nice solution. This is the topic of this paper.

The obtained optimal solution H^* has two features that deserve to be mentioned.

- The larger the accumulated capital at time n is, the larger the position is to be taken (i.e. H_n^*). This is similar to what is done in practice, where the current capital serves as a risk limit. However, H_n^* is not exactly a multiple of the current wealth.
- There exists a stopping time τ such that $H_n^* \neq 0$ on $\{n \le \tau\}$ and $H_n^* = 0$ on $\{n > \tau\}$ (we have $\mathsf{P}(\tau < N) > 0$ and $\mathsf{P}(\tau = N) > 0$). The time τ occurs when a big loss within one step happens. This is very similar to stop-loss limits imposed in practice: if a trader's portfolio suffers a big loss, the position is liquidated and he/she is suspended from trading for some "cooling off" period.

Structure of the paper. In Section 2, we recall basic definitions and facts related to discrete-time coherent risks.

In Section 3, we reduce (1.2)–(1.3) to a static optimization problem with a static risk measure. The class of models we are considering is rather wide: in particular, it includes the conditionally Gaussian models. The class of risk measures considered is the dynamic Weighted V@R, which is a wide subclass of dynamic coherent risks.

In Section 4, we study the static problem, to which (1.2)–(1.3) is reduced. For the case of a conditionally Gaussian model, we reduce the multidimensional problem to the one-dimensional. For the latter one, we present an explicit solution.

Finally, in Section 5, we study the asymptotic behavior of the solution of (1.2)–(1.3) as $N \to \infty$. This is motivated by the fact that in practice there exists no terminal date for the optimization.

2 Coherent Risks

Static risks. Throughout the paper, it will be more convenient to deal not with coherent risks but rather with their opposites — coherent utility functions (this enables one to get rid of numerous minus signs). Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. According to the definition proposed in [3], [4], [16], a *coherent utility* is a map $u : L^\infty \to \mathbb{R}$ satisfying certain axioms. According to the representation theorem established in [4] for a finite Ω and in [16] for a general case (in this case an additional axiom called the Fatou property is needed), a map u is a coherent utility if and only if it can be represented in the form

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X \tag{2.1}$$

with some set $\mathcal{D}$ of probability measures absolutely continuous with respect to P. From the financial point of view, X is the P&L of some portfolio.

For financial applications one needs to extend coherent utilities to unbounded random variables (indeed, most natural distributions like the normal one are unbounded). This can be achieved simply by defining a coherent utility on the space L^0 of all random variables as a map of the form (2.1), where $\mathcal{D}$ is a set of measures absolutely continuous with respect to P and the expectation $\mathsf{E}_\mathsf{Q} X$ is understood in the generalized sense: $\mathsf{E}_\mathsf{Q} X = \mathsf{E}_\mathsf{Q} X^+ - \mathsf{E}_\mathsf{Q} X^-$ with the convention $\infty - \infty = -\infty$. (Throughout the paper, all the expectations are understood in this way.) This way of defining coherent risks on L^0 proved to be very convenient (see [11], [12]).

Throughout the paper, we identify probability measures that are absolutely continuous with respect to P (these are typically denoted as Q) with their densities with respect to P (these are typically denoted as Z); E without a subscript means the expectation with respect to the original measure P.

A basic example of coherent risks is *Tail V@R*. This is a coherent utility u_λ corresponding to

$$\mathcal{D} = \mathcal{D}_\lambda = \{Z : 0 \le Z \le \lambda^{-1}, \mathsf{E}Z = 1\},$$

where $\lambda \in (0, 1]$. If X has a continuous distribution, then

$$u_\lambda(X) = \mathsf{E}(X \,|\, X \le q_\lambda),$$

where q_λ is a λ-quantile of X.

Another basic example is *Weighted V@R* defined as

$$u_\mu(X) = \int_{(0,1]} u_\lambda(X)\mu(d\lambda), \quad X \in L^0, \tag{2.2}$$

where μ is a probability measure on $(0,1]$ and the integral $\int f(\lambda)\mu(d\lambda)$ is understood as $\int f^+(\lambda)\mu(d\lambda) - \int f^-(\lambda)\mu(d\lambda)$ with the convention $\infty - \infty = -\infty$. It can be checked that u_μ is a coherent utility, i.e.

$$u_\mu(X) = \inf_{\mathsf{Q}\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0 \tag{2.3}$$

with

$$\mathcal{D} = \mathcal{D}_\mu = \{Z : Z \geq 0,\ \mathsf{E}Z = 1,\ \mathsf{E}(Z-x)^+ \leq \Phi_\mu(x)\ \forall x \in \mathbb{R}_+\}, \tag{2.4}$$

where

$$\Phi_\mu(x) = \sup_{y\in[0,1]} \biggl[\int_0^y \int_z^1 \lambda^{-1}\mu(d\lambda)dz - xy\biggr], \quad x \in \mathbb{R}_+ \tag{2.5}$$

(see [10; Th. 4.6]).

Dynamic risks. As opposed to the static situation, in the dynamic case there is still no complete unanimity about the right definition of a coherent risk (see the review and the discussion in [13]). However, at least in the discrete-time case many researchers agree on a single definition. It was proposed independently by Cheridito, Delbaen, and Kupper [7] and Jobert, Rogers [24]. The corresponding representation theorem was established by Cheridito and Kupper [9]. All these papers deal with bounded processes. In order to extend dynamic coherent risks to the unbounded case, we proceed as in the static case: the representation is taken as the definition of a coherent risk for unbounded processes. Below we recall the corresponding definition taken from [13].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$ be a discrete-time filtered probability space. Let $(\mathcal{D}_n)_{n=1,\dots,N}$ be a system of sets of random variables with the properties:

- any random variable Z from $\mathcal{D}_n$ is positive, $\mathcal{F}_n$-measurable, and satisfies the inequality $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) \leq 1$;
- $\mathcal{D}_n$ is non-empty, L^1-closed, uniformly integrable, and $\mathcal{F}_{n-1}$*-convex*, i.e. for any $Z_1, Z_2 \in \mathcal{D}_n$ and any $[0,1]$-valued $\mathcal{F}_{n-1}$-measurable random variable λ, we have $\lambda Z_1 + (1-\lambda)Z_2 \in \mathcal{D}_n$.

Let $X = (X_n)_{n=0,\dots,N}$ be a one-dimensional $(\mathcal{F}_n)$-adapted process. From the financial point of view, X describes the stream of cash flows of some portfolio, i.e. X_n is the discounted amount received at time n (for example, the stream of cash flows corresponding to a strategy H in (1.3) is given by $X_n = \sum_i H_n^i(S_n^i - S_{n-1}^i)$). The *coherent utility* of X is the $[-\infty,\infty]$-valued process $u(X) = (u_n(X))_{n=0,\dots,N}$ defined as: $u_N(X) = 0$,

$$u_{n-1}(X) = \operatorname*{essinf}_{Z\in\mathcal{D}_n} \mathsf{E}(Z(X_n + u_n(X))\,|\,\mathcal{F}_{n-1}), \quad n = N,\dots,1, \tag{2.6}$$

where $\mathsf{E}(\xi\,|\,\mathcal{G})$ is understood as $\mathsf{E}(\xi^+\,|\,\mathcal{G}) - \mathsf{E}(\xi^-\,|\,\mathcal{G})$ with the convention $\infty - \infty = -\infty$; for a $[0,\infty]$-valued random variable ζ, the conditional expectation $\mathsf{E}(\zeta\,|\,\mathcal{G})$ is understood as $\lim_n \mathsf{E}(\zeta \wedge n\,|\,\mathcal{G})$. (Throughout the paper, all the conditional expectations are understood in this way.) From the financial point of view, $-u_n(X)$ is the risk contained in $(X_{n+1},\dots,X_N)$ and measured at time n. Informally, (2.6) reads as follows: the risk of the part of the stream remaining after time $n-1$ is the risk of the cash flow received at time n plus the risk of the part remaining after time n.

If additionally we assume that $\mathsf{E}(Z \mid \mathcal{F}_{n-1}) = 1$ for every n and every $Z \in \mathcal{D}_n$, while X satisfies certain integrability conditions, then the coherent utility gets a simpler representation:

$$u_n(X) = \operatorname*{essinf}_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}\biggl[\sum_{k=n+1}^{N} X_k \biggm| \mathcal{F}_n\biggr], \quad n = 0,\dots,N,$$

where $\mathcal{D} = \bigl\{\prod_{n=1}^N Z_n : Z_n \in \mathcal{D}_n\bigr\}$ (see [13; Prop. 2.2]). However, it is important to consider coherent risks, for which $\mathsf{E}(Z \mid \mathcal{F}_{n-1})$ might be strictly smaller than 1. This corresponds to the effect that might be called the discounting of risk: the risk of losses in the far future is encountered with a smaller weight than the risk of losses in the near future. As an example, let $\mathcal{D}_n$ be such that $\mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) = \alpha$ for any n and any $Z \in \mathcal{D}_n$, where $\alpha \in (0,1]$; consider X of the form $X_n = 0$ for $n < N$ and $X_N = -1$ (an obligation to pay 1 unit of money at time N). Then $u_0(X) = -\alpha^N$, i.e. the risk of X at time 0 is α^N. Note that this has nothing to do with the ordinary discounting as X_n are already discounted values.

The natural dynamic analog of Tail V@R is obtained by taking

$$\mathcal{D}_n = \{Z : Z \text{ is } \mathcal{F}_n\text{-measurable, } 0 \le Z \le \lambda^{-1}, \text{ and } \mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) = 1\}, \quad n = 1,\dots,N.$$

Using representation (2.3)–(2.5), we can also extend Weighted V@R to the dynamic case by setting

$$\begin{aligned}\mathcal{D}_n = \{Z : Z &\text{ is } \mathcal{F}_n\text{-measurable, } Z \ge 0,\, \mathsf{E}(Z\,|\,\mathcal{F}_{n-1}) = 1,\\ &\text{and } \mathsf{E}((Z-x)^+\,|\,\mathcal{F}_{n-1}) \le \Phi_\mu(x)\ \forall x \in \mathbb{R}_+\}, \quad n = 1,\dots,N.\end{aligned} \tag{2.7}$$

Clearly, the sets $\mathcal{D}_n$ satisfy the conditions imposed above. As shown in [13; Lem. 2.5]), equality (2.6) is then rewritten as

$$u_{n-1}(X) = \widetilde{u}_\mu(\mathsf{Law}(X_n + u_n(X)\,|\,\mathcal{F}_{n-1})), \tag{2.8}$$

where $\widetilde{u}_\mu$ is a map defined on distributions such that $u_\mu(\xi) = \widetilde{u}_\mu(\mathsf{Law}\,\xi)$ for any random variable ξ (such a map exists because u_λ and hence, u_μ depend only on the distribution of a random variable). This shows the relevance of the given definition (2.7).

Let us finally remark that the functional u_μ can be defined by (2.2) also for a positive (not necessarily probabilistic) finite measure μ on $(0,1]$, and representation (2.3)–(2.5) remains valid with the condition $\mathsf{E}Z = 1$ replaced by $\mathsf{E}Z = \mu((0,1])$. This enables us to extend the dynamic Weighted V@R to measures μ with $\mu((0,1]) \le 1$ simply by replacing in (2.7) the condition $\mathsf{E}(Z \,|\, \mathcal{F}_{n-1}) = 1$ by the condition $\mathsf{E}(Z \,|\, \mathcal{F}_{n-1}) = \mu((0,1])$.

3 Dynamic Optimization Problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,\dots,N}, \mathsf{P})$ be a filtered probability space. Let $(S_n)_{n=0,\dots,N}$ be a d-dimensional process of the form

$$S_n = S_0 + \sum_{i=1}^{n} \sigma_i X_i, \tag{3.1}$$

where X_n is a d-dimensional integrable $\mathcal{F}_n$-measurable random vector independent of $\mathcal{F}_{n-1}$, all X_n are identically distributed, and σ_n is a non-degenerate $\mathcal{F}_{n-1}$-measurable $d \times d$-matrix. From the financial point of view, we have a market consisting of d traded assets and S_n^i is the discounted price of the i-th asset at time n. Note that any conditionally Gaussian model belongs to this class (provided that the covariance matrices are non-degenerate and integrable). In particular, multidimensional GARCH models (see, for example, [15; § 3.4]) are of this form.

Let μ be a positive measure on $(0,1]$ with $\mu((0,1]) \le 1$ and $\mu((0,1)) > 0$. Let $(u_n)_{n=0}^N$ be the corresponding dynamic Weighted V@R. We assume that $\mathsf{E}X_n \ne 0$ and

$$-\infty < u_\mu(\langle h, X_n \rangle) < 0 \quad \forall h \in \mathbb{R}^d \setminus \{0\}, \tag{3.2}$$

where u_μ is given by (2.2) and $\langle \cdot\,, \cdot \rangle$ denotes the scalar product. The first condition means that there exists a trade with strictly positive reward; the second one means that any simple trade has finite strictly positive risk.

Let $\mathcal{A}$ denote the space of d-dimensional predictable processes. From the financial point of view, a process $H \in \mathcal{A}$ is a dynamic trading strategy, i.e. H_n^i is the position in the i-th asset taken at time $n-1$, so that the corresponding discounted cash flow received at time n is $\sum_i H_n^i \Delta S_n^i = \langle H_n, \Delta S_n \rangle$, where $\Delta S_n = S_n - S_{n-1}$. Let $\alpha \in (0,1]$ be the coefficient meaning the time value of money, i.e. the value at time 0 of an amount x received at time n is $\alpha^n x$. This is not the discounting because x is already a discounted amount, but rather the expression of the fact that getting something today is better than getting it 10 years later.

We will consider the problem

$$\begin{cases} \mathsf{E}\sum_{n=1}^{N}\alpha^n\langle H_n,\Delta S_n\rangle \longrightarrow \max, \\ H\in\mathcal{A}, \\ u_0(\langle H,\Delta S\rangle)\ge -c, \end{cases} \tag{3.3}$$

where $\langle H,\Delta S\rangle = (\langle H_n,\Delta S_n\rangle)_{n=1,\dots,N}$ is the stream of cash flows corresponding to a strategy H and c is a strictly positive constant.

In order to describe the optimal solution, we introduce some notation. Consider the set

$$\mathcal{H} = \{h\in\mathbb{R}^d : u_\mu(\langle h,\bar{X}\rangle)\ge -1\},$$

where $\bar{X}$ is a random vector that coincides in distribution with all X_n ($\mathcal{H}$ is defined correctly because $u_\mu(\xi)$ depends only on the distribution of ξ). Due to (3.2), $\mathcal{H}$ is a convex compact. Denote

$$a(h) = \inf\{x\in\mathbb{R} : u_\mu(\langle h,\bar{X}\rangle\wedge x)\ge -1\},\quad h\in\mathcal{H}, \tag{3.4}$$

where $\inf\emptyset = +\infty$ (thus, $-\mu((0,1])^{-1}\le a(h)\le\infty$)). Construct numbers $(R_n^*)_{n=0,\dots,N}$ and vectors $(h_n^*)_{n=0,\dots,N-1}$ going backwards from N to 0 by: $R_N^*=0$,

$$R_{n-1}^* = \max_{h\in\mathcal{H}}\alpha\mathsf{E}\big[\langle h,\bar{X}\rangle + R_n^*(\langle h,\bar{X}\rangle - a(h))^+\big],\quad n=N,\dots,1, \tag{3.5}$$

$$h_{n-1}^* = \operatorname*{argmax}_{h\in\mathcal{H}}\mathsf{E}\big[\langle h,\bar{X}\rangle + R_n^*(\langle h,\bar{X}\rangle - a(h))^+\big],\quad n=N,\dots,1. \tag{3.6}$$

Note that R_n^* is finite and h_n^* exists due to the compactness of $\mathcal{H}$. Construct a d-dimensional predictable process H^* and a one-dimensional adapted process C^* going forwards from 0 to N by: $C_0^*=c$,

$$H_n^* = C_{n-1}^*(\sigma_n^t)^{-1}h_{n-1}^*,\quad n=1,\dots,N,$$
$$C_n^* = C_{n-1}^*(\langle h_{n-1}^*,X_n\rangle - a(h_{n-1}^*))^+,\quad n=1,\dots,N,$$

where σ_n^t is the transpose of σ_n.

Theorem 3.1. *The optimal value in* (3.3) *equals* R_0^*c, *and* H^* *is an optimal strategy. If moreover each* h_n^* *is unique, then the optimal strategy is unique.*

To illustrate the optimal solution we have described, consider the following example. Let $N=2$, $d=1$, $c=1$, $\sigma=1$, $\alpha=1$, and $\mathcal{F}_0$ be trivial. We assume that $\mathsf{E}X_n>0$. Then, for any $\mathcal{F}_1$-measurable random variable H_2, we have

$$\mathsf{E}(H_2X_2\,|\,\mathcal{F}_1) = H_2\mathsf{E}X_2,$$
$$u_1(HX) = H_2u_\mu(X_2).$$

Thus,

$$\mathsf{E}(H_2X_2 \,|\, \mathcal{F}_1) = -R_1^* u_1(HX) = -R_1^* H_2 u_\mu(X_2),$$

where $R_1^* = -\mathsf{E}X_2/u_\mu(X_2)$ (clearly, it coincides with the value given by (3.5)). Furthermore, for a fixed $H_1 = h \in \mathbb{R}$,

$$u_0(HX) = u_\mu(hX_1 + u_1(HX)).$$

Denote hX_1 by ζ and $hX_1 + u_1(HX)$ by η. Then

$$\mathsf{E}(H_1X_1 + H_2X_2) = \mathsf{E}(hX_1 - R_1^* u_1(HX)) = (1 + R_1^*)\mathsf{E}hX_1 - R_1^*\mathsf{E}\eta,$$

while the condition $u_0(HX) \ge -1$ becomes $u_\mu(\eta) \ge -1$. So, for a fixed H_1, the choice of the optimal H_2 is the problem

$$\begin{cases} \mathsf{E}\eta \longrightarrow \min, \\ \eta \le \zeta, \\ u_\mu(\eta) \ge -1 \end{cases}$$

(note that $u_1(HX) \le 0$ due to assumption (3.2)). If $u_\mu(\zeta) < -1$, then there exists no solution; if $u_\mu(\zeta) \ge -1$, then, as shown by Lemma 3.2 below, the solution of this problem has the form $\eta^* = \zeta \wedge a$, where a is a constant such that $u_\mu(\zeta \wedge a) = -1$. Thus, $a = a(h)$ in our notation. We see that, under a fixed $H_1 = h$, the optimal H_2 is given by the equation

$$-H_2^* u_\mu(X_2) = \zeta - \eta^* = (hX_1 - a(h))^+.$$

Then

$$\mathsf{E}(H_1X_1 + H_2^*X_2) = \mathsf{E}(hX_1 - R_1^* H_2^* u_\mu(X_2)) = \mathsf{E}(hX_1 + R_1^*(hX_1 - a(h))^+).$$

The optimal H_1^* is found as the solution of the problem

$$\mathsf{E}(hX_1 + R_1^*(hX_1 - a(h))^+) \longrightarrow \max,$$

where h runs through $\{h : u_\mu(hX_1) \ge -1\} = \mathcal{H}$. The optimal value in this problem is R_0^* and the optimal h is h_0^*. Finally, the optimal value in the original problem (3.3) is R_0^*, and the optimal strategy is given by $H_1^* = h_0^*$, $H_2^* = -(u_\mu(X_2))^{-1}(h_0^*X_1 - a(h_0^*))^+$. This coincides with the optimal solution described above, as in our example $C_0^* = 1$, $C_1^* = (h_0^*X_1 - a(h_0^*))^+$, and $h_1^* = -(u_\mu(X_2))^{-1}$. It is seen that the process C^* has the meaning of risk allowed to be taken in the future, i.e. C_n^* is the risk allowed to be taken after time n.

`Remark`. Consider the stopping time

$$\tau = \inf\{n : \langle h_{n-1}^*, X_n \rangle \le a(h_{n-1}^*)\}.$$

Then $H_n^* \neq 0$ on $\{n \le \tau\}$ and $H_n^* = 0$ on $\{n > \tau\}$. Thus, the optimal strategy has the property: trading is switched off after a big loss has occurred.

The proof of Theorem 3.1 is broken into a series of lemmas.

Lemma 3.2. *Let $c > 0$ and ζ be an integrable random variable with $u_\mu(\zeta) \ge -c$. Denote $a = \inf\{x \in \mathbb{R} : u_\mu(\zeta \wedge x) \ge -c\}$ and $\eta^* = \zeta \wedge a$. Then, for any $\eta \le \zeta$ such that $u_\mu(\eta) \ge -c$, we have $\mathsf{E}\eta \ge \mathsf{E}\eta^*$. Moreover, η^* is the unique random variable with this property.*

`Proof`. We can assume that the probability space is atomless (if not, we extend it to an atomless space).

Step 1. Suppose that there exists η such that $\eta \le \zeta$, $u_\mu(\eta) \ge -c$, and $\mathsf{E}\eta < \mathsf{E}\eta^*$. Then, for any integrable random variable ξ and any $\lambda \in (0,1]$, there exists a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\xi) = \mathsf{E}(\xi \,|\, D)$ (to construct it, take a λ-quantile q of ξ and set $D = \{\xi < q\} \cup \bar{D}$, where $\bar{D} \subseteq \{\xi = q\}$ and $\mathsf{P}(\bar{D}) = \lambda - \mathsf{P}(\xi < q)$). Note also that, for any integrable random variable ξ and any set D' with $\mathsf{P}(D') = \lambda$, we have $u_\lambda(\xi) \le \mathsf{E}(\xi \,|\, D')$. Denote $A = \{\zeta < a\}$ and $s = \sup\{x : x \in \operatorname{supp} \mu\}$, where "supp" denotes the support.

Case 1. Suppose that $\mathsf{P}(A) = 1$. This implies that $u_\mu(\zeta) = -c$ and $s = 1$ since otherwise we can take $a' < a$ such that $u_\mu(\zeta \wedge a') \ge -c$. We can find $b < a$ such that $\mathsf{P}(B) < 1$ and $\mathsf{P}(B \cap \{\eta < \zeta\}) > 0$, where $B = \{\zeta < b\}$. Fix $\lambda \in (\mathsf{P}(B), 1]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\zeta) = \mathsf{E}(\zeta \mid D)$. Clearly, $D \supseteq B$. Hence, $u_\lambda(\eta) \le \mathsf{E}(\eta \mid D) < \mathsf{E}(\zeta \mid D) = u_\lambda(\zeta)$. As this is true for any $\lambda \in (\mathsf{P}(B), 1]$ and $\mu((\mathsf{P}(B), 1]) > 0$ (this follows from the equality $s = 1$), while $u_\lambda(\eta) \le u_\lambda(\zeta)$ for any λ, we get $u_\mu(\eta) < u_\mu(\zeta) = -c$, which is a contradiction.

Case 2. Suppose that $\mathsf{P}(A) < 1$. Set $\eta_1 = \mathsf{E}(\eta \,|\, \mathcal{G}(A^c))$, where $\mathcal{G}(A^c)$ is the largest sub-σ-field of $\mathcal{F}$ containing A^c as an atom. It is easy to see that $u_\lambda(\eta) \le u_\lambda(\eta_1)$ for any $\lambda \in (0,1]$, so that $u_\mu(\eta) \le u_\mu(\eta_1)$.

Let η_2 be the random variable that is equal to η on A and equals a constant on A^c, where the constant is chosen in such a way that $\mathsf{E}\eta_2 = \mathsf{E}\eta^*$. Then $\eta_1 \le \eta_2$, so that $u_\mu(\eta_1) \le u_\mu(\eta_2)$.

Take $\lambda \in (0, \mathsf{P}(A)]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\eta^*) = \mathsf{E}(\eta^* \mid D)$. Clearly, $D \subseteq A$. Hence, $\eta_2 = \eta \le \zeta = \eta^*$ on D, so that $u_\lambda(\eta_2) \le \mathsf{E}(\eta_2 \,|\, D) \le \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$.

Take now $\lambda \in (\mathsf{P}(A), 1]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\eta^*) = \mathsf{E}(\eta^* \,|\, D)$. Clearly, $D \supseteq A$. Note that $\eta_2 \le \eta^*$ on A, $\eta_2 \ge \eta^*$ on A^c, and $\mathsf{E}\eta_2 = \mathsf{E}\eta^*$. Hence, $u_\lambda(\eta_2) \le \mathsf{E}(\eta_2 \,|\, D) \le \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$. As a result, $u_\mu(\eta_2) \le u_\mu(\eta^*)$.

Case 2.1. Suppose that $s > \mathsf{P}(A)$. Take $\lambda \in (\mathsf{P}(A), 1]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\eta_2) = \mathsf{E}(\eta_2 \,|\, D)$. As $\eta_1 = \eta_2 = \eta$ on A and $\eta_1 < \eta_2$ on A^c (this follows from the inequality $\mathsf{E}\eta < \mathsf{E}\eta^*$), we have $u_\lambda(\eta_1) \le \mathsf{E}(\eta_1 \,|\, D) < \mathsf{E}(\eta_2 \,|\, D) = u_\lambda(\eta_2)$. As this is true for any $\lambda \in (\mathsf{P}(A), 1]$ and $\mu((\mathsf{P}(A), 1]) > 0$ (this follows from the inequality $s > \mathsf{P}(A)$), while $u_\lambda(\eta_1) \le u_\lambda(\eta_2)$ for any λ, we get $u_\mu(\eta_1) < u_\mu(\eta_2)$.

Case 2.2. Suppose that $s \le \mathsf{P}(A)$. Then $s = \mathsf{P}(A)$ since otherwise we can find $a' < a$ such that $u_\mu(\zeta \wedge a') \ge -c$. We can find $b < a$ such that $\mathsf{P}(B) < \mathsf{P}(A)$ and $\mathsf{P}(B \cap \{\eta < \zeta\}) > 0$, where $B = \{\zeta < b\}$. As $\eta_2 = \eta$ and $\eta^* = \zeta$ on A, we have $\mathsf{P}(B \cap \{\eta_2 < \eta^*\}) > 0$. Fix $\lambda \in (\mathsf{P}(B), \mathsf{P}(A)]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\eta^*) = \mathsf{E}(\eta^* \,|\, D)$. Clearly, $D \supseteq B$. Hence, $u_\lambda(\eta_2) \le \mathsf{E}(\eta_2 \,|\, D) < \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$. As this is true for any $\lambda \in (\mathsf{P}(B), \mathsf{P}(A)]$ and $\mu((\mathsf{P}(B), \mathsf{P}(A)]) > 0$ (this follows from the equality $s = \mathsf{P}(A)$), while $u_\lambda(\eta_2) \le u_\lambda(\eta^*)$ for any λ, we get $u_\mu(\eta_2) < u_\mu(\eta^*)$.

As a result, in both cases 2.1 and 2.2 we get $u_\mu(\eta) < u_\mu(\eta^*) = -c$, which is a contradiction.

Step 2. Let us prove the uniqueness part. Suppose that there exists $\eta \le \zeta$ such that $u_\mu(\eta) \ge -c$, $\mathsf{E}\eta = \mathsf{E}\eta^*$, and $\eta \ne \eta^*$. For the measure $\widetilde{\mu} = \mu\big|_{(0,1)}$ and the constant $\widetilde{c} = c\mu((0,1))/\mu((0,1])$, we have $u_{\widetilde{\mu}}(\eta) = u_{\widetilde{\mu}}(\eta^*) = -\widetilde{c}$, so we can assume from the outset that $\mu(\{1\}) = 0$. Let A and s be the same as above.

Case 1. Suppose that $\mathsf{P}(A) = 1$. This means that $\eta^* = \zeta$. But then the inequality $\eta \le \zeta$ and the equality $\mathsf{E}\eta = \mathsf{E}\eta^*$ imply that $\eta = \eta^*$, which is a contradiction.

Case 2. Suppose that $\mathsf{P}(A) < 1$. In this case $a < \infty$ and $s \ge \mathsf{P}(A)$ since otherwise we can take $a' < a$ such that $u_\mu(\zeta \wedge a') \ge -c$.

Case 2.1. Suppose that $\mathsf{P}(A \cap \{\eta < \zeta\}) > 0$. We can find $b < a$ such that $\mathsf{P}(B) < \mathsf{P}(A)$ and $\mathsf{P}(B \cap \{\eta < \zeta\}) > 0$, where $B = \{\zeta < b\}$.

Take $\lambda \in (0, \mathsf{P}(B)]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\eta^*) = \mathsf{E}(\eta^* \,|\, D)$. Clearly, $D \subseteq B$. Hence, $\eta \le \zeta = \eta^*$ on D, so that $u_\lambda(\eta) \le \mathsf{E}(\eta \,|\, D) \le \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$.

Take $\lambda \in (\mathsf{P}(B), \mathsf{P}(A)]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\eta^*) = \mathsf{E}(\eta^* \,|\, D)$. Clearly, $B \subseteq D \subseteq A$, so that $u_\lambda(\eta) \le \mathsf{E}(\eta \,|\, D) < \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$.

Take $\lambda \in (\mathsf{P}(A), 1)$. Since $\eta \le \eta^*$ on A, $\mathsf{P}(A^c \cap \{\eta > \eta^*\}) > 0$, and $\mathsf{E}\eta = \mathsf{E}\eta^*$, we can find a set D such that $\mathsf{P}(D) = \lambda$, $D \supseteq A$, and $\mathsf{E}(\eta \,|\, D) < \mathsf{E}(\eta^* \,|\, D)$. As $\eta^* < a$ on A and $\eta^* = a$ on A^c, we have $u_\lambda(\eta^*) = \mathsf{E}(\eta^* \,|\, D)$. Thus, $u_\lambda(\eta) \le \mathsf{E}(\eta \,|\, D) < \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$.

As a result, $u_\lambda(\eta) < u_\lambda(\eta^*)$ for any $\lambda \in (\mathsf{P}(B), 1)$. As $\mu(\mathsf{P}(B), 1) > 0$ (this follows from the inequality $s \ge \mathsf{P}(A)$), while $u_\lambda(\eta) \le u_\lambda(\eta^*)$ for any λ, we get $u_\mu(\eta) < u_\mu(\eta^*) = -c$, which is a contradiction.

Case 2.2. Suppose that $\eta = \zeta$ on A. Then $\mathsf{P}(A^c \cap \{\eta < a\}) > 0$. We can find $b < a$ such that $\mathsf{P}(B) < \mathsf{P}(A)$ and $\mathsf{P}(A^c \cap \{\eta < b\}) > 0$, where $B = \{\zeta < b\}$.

For any $\lambda \in (0, \mathsf{P}(B))$, we have $u_\lambda(\eta) \le u_\lambda(\eta^*)$, which is proved in the same way as above.

Take $\lambda \in (\mathsf{P}(B), \mathsf{P}(A)]$ and find a set D with $\mathsf{P}(D) = \lambda$ such that $u_\lambda(\eta^*) = \mathsf{E}(\eta^* \,|\, D)$.

Clearly, $B \subseteq D \subseteq A$ and $\mathsf{P}(D \setminus B) > 0$. We can find a set D' with $\mathsf{P}(D') = \mathsf{P}(D)$ that is obtained from D by removing a subset of $D \setminus B$ and adding a subset of $A^c \cap \{\eta < b\}$. As $\eta = \zeta = \eta^* \geq b$ on $D \setminus B$, we get $\mathsf{E}(\eta \mid D') < \mathsf{E}(\eta \mid D) = \mathsf{E}(\eta^* \mid D)$. Hence, $u_\lambda(\eta) \leq \mathsf{E}(\eta \,|\, D') < \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$.

Take $\lambda \in (\mathsf{P}(A), 1)$. As $\eta = \eta^*$ on A and $\mathsf{E}\eta = \mathsf{E}\eta^*$, we have $\mathsf{E}(\eta \mid A^c) = \mathsf{E}(\eta^* \mid A^c) = \mathsf{E}(a \mid A^c)$. Moreover, $\mathsf{P}(A^c \cap \{\eta \neq a\}) > 0$, so that we can find a set D with $\mathsf{P}(D) = \lambda$ such that $D \supseteq A$ and $\mathsf{E}(\eta \,|\, D \cap A^c) < \mathsf{E}(\eta^* \,|\, D \cap A^c)$. Then $u_\lambda(\eta) \leq \mathsf{E}(\eta \,|\, D) < \mathsf{E}(\eta^* \,|\, D) = u_\lambda(\eta^*)$.

As a result, $u_\lambda(\eta) < u_\lambda(\eta^*)$ for any $\lambda \in (\mathsf{P}(B), 1)$. As $\mu(\mathsf{P}(B), 1) > 0$ (this follows from the inequality $s \geq \mathsf{P}(A)$), while $u_\lambda(\eta) \leq u_\lambda(\eta^*)$ for any λ, we get $u_\mu(\eta) < u_\mu(\eta^*) = -c$, which is a contradiction. □

Below we denote by X the process $(X_n)_{n=1}^N$; by $\langle H, X\rangle$ we denote the process $(\langle H_n, X_n\rangle)_{n=1}^N$.

Lemma 3.3. *For any $H \in \mathcal{A}$, we have*

$$u_n(\langle H, X\rangle) \leq 0, \quad n = 0, \dots, N.$$

Proof. We prove the statement going backwards from N to 0. For $n = N$, it is trivial. Suppose the statement is true for n, and let us prove it for $n-1$. It follows from (2.8) that, for a.e. ω,

$$u_{n-1}(\langle H, X\rangle)(\omega) \leq \operatorname*{essinf}_{Z \in \mathcal{D}_n} \mathsf{E}(Z\langle H_n, X_n\rangle \,|\, \mathcal{F}_{n-1})(\omega) = u_\mu(\langle h, \bar{X}\rangle),$$

where $h = H_n(\omega)$ and $\bar{X}$ is a random vector having the same distribution as all X_n. Now, assumption (3.2) ensures that $u_{n-1}(\langle H, X\rangle) \leq 0$. □

Lemma 3.4. *For any random variable ξ and σ-fields $\mathcal{G}_1 \subseteq \mathcal{G}_2$, we have*

$$\mathsf{E}(\mathsf{E}(\xi \,|\, \mathcal{G}_2) \,|\, \mathcal{G}_1) \geq \mathsf{E}(\xi \,|\, \mathcal{G}_1),$$

where the conditional expectation is understood in the generalized sense.

Proof. Denote $\mathsf{E}(\xi \,|\, \mathcal{G}_2)^+$ by ζ and $\mathsf{E}(\xi \,|\, \mathcal{G}_2)^-$ by η. Then

$$\mathsf{E}(\mathsf{E}(\xi \,|\, \mathcal{G}_2) \,|\, \mathcal{G}_1) = \mathsf{E}(\zeta - \eta \,|\, \mathcal{G}_1) \geq \mathsf{E}(\zeta \,|\, \mathcal{G}_1) - \mathsf{E}(\eta \,|\, \mathcal{G}_1) = \mathsf{E}(\xi^+ \,|\, \mathcal{G}_1) - \mathsf{E}(\xi^- \,|\, \mathcal{G}_1) = \mathsf{E}(\xi \,|\, \mathcal{G}_1).$$

The inequality above is proved as follows: on the set $\{\mathsf{E}(\eta \mid \mathcal{G}_1) = \infty\}$ it is trivial as its right-hand side is $-\infty$; the set $\{\mathsf{E}(\eta \mid \mathcal{G}_1) < \infty\}$ is represented as the union of sets $\{\mathsf{E}(\eta \,|\, \mathcal{G}_1) \leq n\}$, on each of which the inequality becomes the equality. □

Lemma 3.5. *For any $H \in \mathcal{A}$, we have*

$$\mathsf{E}\left[\sum_{k=n+1}^{N} \alpha^{k-n}\langle H_k, X_k\rangle \,\middle|\, \mathcal{F}_n\right] \le -R_n^* u_n(\langle H, X\rangle), \quad n = 0,\dots,N.$$

`Proof`. We will prove the statement going backwards from N to 0. For $n = N$, it is trivial. Suppose it is true for n, and let us prove it for $n-1$. In view of (2.8), the equality

$$u_{n-1}(\langle H, X\rangle) = \operatorname*{essinf}_{\mathsf{Q}\in\mathcal{D}_n} \mathsf{E}_{\mathsf{Q}}(\langle H_n, X_n\rangle + u_n(\langle H, X\rangle) \,|\, \mathcal{F}_{n-1})$$

means that, for a.e. ω,

$$v = u_\mu(\langle h, \widetilde{X}\rangle + \widetilde{V}), \tag{3.7}$$

where $v = u_{n-1}(\langle H, X\rangle)(\omega)$, $h = H_n(\omega)$, and $(\widetilde{X}, \widetilde{V})$ is a random vector with $\mathsf{Law}(\widetilde{X}, \widetilde{V}) = \mathsf{Law}(X_n, u_n(\langle H, X\rangle) \,|\, \mathcal{F}_{n-1})(\omega)$ (it is not important on which space $(\widetilde{X}, \widetilde{V})$ is defined as $u_\mu(\xi)$ depends only on the distribution of ξ). As X_n is independent of $\mathcal{F}_{n-1}$, we have $\mathsf{Law}\,\widetilde{X} = \mathsf{Law}\,X_n$. Let us prove that

$$\alpha\mathsf{E}(\langle h, \widetilde{X}\rangle - R_n^*\widetilde{V}) \le -R_{n-1}^* v. \tag{3.8}$$

If $v = -\infty$, then (3.8) is trivial. If $v = 0$, then, using equality (3.7), assumption (3.2), and Lemma 3.3, we deduce that $h = 0$ and $\widetilde{V} = 0$, in which case (3.8) is again trivial. Now, let $-\infty < v < 0$. Denote $\overline{h} = h/|v|$ and $\overline{V} = \widetilde{V}/|v|$. Then $\overline{V} \le 0$ due to Lemma 3.3 and $u_\mu(\langle \overline{h}, \widetilde{X}\rangle + \overline{V}) = -1$ due to (3.7). It follows from Lemma 3.2 applied to $X = \langle \overline{h}, \widetilde{X}\rangle$ and $Y = \langle \overline{h}, \widetilde{X}\rangle + \overline{V}$ that

$$-\mathsf{E}\overline{V} \le \mathsf{E}(X - Y) \le \mathsf{E}(X - X \wedge a(\overline{h})) = \mathsf{E}(X - a(\overline{h}))^+,$$

so that

$$\alpha\mathsf{E}(\langle \overline{h}, \widetilde{X}\rangle - R_n^*\overline{V}) \le \alpha\mathsf{E}(\langle \overline{h}, \widetilde{X}\rangle - R_n^*(\langle \overline{h}, \widetilde{X}\rangle - a(\overline{h}))^+) \le R_{n-1}^*.$$

Thus, (3.8) is proved. Then we can write

$$\begin{aligned}
&\mathsf{E}\left[\sum_{k=n}^{N} \alpha^{k-n+1}\langle H_k, X_k\rangle \,\middle|\, \mathcal{F}_{n-1}\right]\\
&= \mathsf{E}(\alpha\langle H_n, X_n\rangle \,|\, \mathcal{F}_{n-1}) + \alpha\mathsf{E}\left[\sum_{k=n+1}^{N} \alpha^{k-n}\langle H_k, X_k\rangle \,\middle|\, \mathcal{F}_{n-1}\right]\\
&\le \mathsf{E}(\alpha\langle H_n, X_n\rangle \,|\, \mathcal{F}_{n-1}) + \alpha\mathsf{E}\left[\mathsf{E}\left[\sum_{k=n+1}^{N} \alpha^{k-n}\langle H_k, X_k\rangle \,\middle|\, \mathcal{F}_n\right] \,\middle|\, \mathcal{F}_{n-1}\right]\\
&\le \mathsf{E}(\alpha\langle H_n, X_n\rangle \,|\, \mathcal{F}_{n-1}) + \alpha\mathsf{E}(-R_n^* u_n(\langle H, X\rangle) \,|\, \mathcal{F}_{n-1})\\
&= \alpha\mathsf{E}(\langle H_n, X_n\rangle - R_n^* u_n(\langle H, X\rangle) \,|\, \mathcal{F}_{n-1})\\
&\le -R_{n-1}^* u_{n-1}(\langle H, X\rangle).
\end{aligned} \tag{3.9}$$

Both equalities here follow from the integrability of X_n and the independence of X_n from $\mathcal{F}_{n-1}$; the first inequality follows from Lemma 3.4; the second inequality follows from the induction assumption; the third inequality follows from (3.8). □

Lemma 3.6. *For any n, C_n^* is integrable.*

Proof. The statement is proved going forwards from 0 to N with the help of the equality

$$\mathsf{E}C_n^* = \mathsf{E}C_{n-1}^* \mathsf{E}(\langle h_{n-1}^*, X_n\rangle - a(h_{n-1}^*))^+,$$

which follows from the independence of X_n and $\mathcal{F}_{n-1}$. □

Lemma 3.7. *We have*

$$u_n(\langle H^*, \Delta S\rangle) \geq -C_n^*, \quad n = 0, \dots, N,$$

$$\mathsf{E}\Bigg[\sum_{k=n+1}^{N} \alpha^{k-n}\langle H_k^*, \Delta S_k\rangle \,\Bigg|\, \mathcal{F}_n\Bigg] = R_n^* C_n^*, \quad n = 0, \dots, N.$$

Proof. We prove the first statement going backwards from 0 to N. For $n = N$, it is trivial. Suppose it is true for $n-1$, and let us prove it for n. We have

$$\begin{aligned}
u_{n-1}(\langle H^*, X\rangle) &= \operatorname*{essinf}_{\mathsf{Q}\in\mathcal{D}_n} \mathsf{E}_{\mathsf{Q}}(\langle H_n^*, X_n\rangle + u_n(\langle H^*, X\rangle) \,|\, \mathcal{F}_{n-1}) \\
&\geq \operatorname*{essinf}_{\mathsf{Q}\in\mathcal{D}_n} \mathsf{E}_{\mathsf{Q}}(C_{n-1}^*\langle h_{n-1}^*, X_n\rangle - C_n^* \,|\, \mathcal{F}_{n-1}) \\
&= C_{n-1}^* \operatorname*{essinf}_{\mathsf{Q}\in\mathcal{D}_n} \mathsf{E}_{\mathsf{Q}}(\langle h_{n-1}^*, X_n\rangle \wedge a(h_{n-1}^*) \,|\, \mathcal{F}_{n-1}) \\
&= C_{n-1}^* u_\mu(\langle h_{n-1}^*, X_n\rangle \wedge a(h_{n-1}^*)) \\
&= -C_{n-1}^*.
\end{aligned}$$

The inequality follows from the induction assumption; the third equality follows from (2.8).

Now, let us prove the second statement going backwards from N to 0. For $n = N$, it is trivial. Suppose it is true for n, and let us prove it for $n-1$. We have

$$\begin{aligned}
&\mathsf{E}\Bigg[\sum_{k=n}^{N} \alpha^{k-n+1}\langle H_n^*, X_n\rangle \,\Bigg|\, \mathcal{F}_{n-1}\Bigg] \\
&= \alpha\mathsf{E}(C_{n-1}^*\langle h_{n-1}^*, X_n\rangle \,|\, \mathcal{F}_{n-1}) + \alpha\mathsf{E}\Bigg[\sum_{k=n+1}^{N} \alpha^{k-n}\langle H_n^*, X_n\rangle \,\Bigg|\, \mathcal{F}_{n-1}\Bigg] \\
&= \alpha\mathsf{E}(C_{n-1}^*\langle h_{n-1}^*, X_n\rangle \,|\, \mathcal{F}_{n-1}) + \alpha\mathsf{E}(R_n^* C_n^* \,|\, \mathcal{F}_{n-1}) \\
&= \alpha C_{n-1}^* \mathsf{E}(\langle h_{n-1}^*, X_n\rangle + R_n^*(\langle h_{n-1}^*, X_n\rangle - a(h_{n-1}^*))^+ \,|\, \mathcal{F}_{n-1}) \\
&= \alpha C_{n-1}^* \mathsf{E}(\langle h_{n-1}^*, X_n\rangle + R_n^*(\langle h_{n-1}^*, X_n\rangle - a(h_{n-1}^*))^+) \\
&= R_{n-1}^* C_{n-1}^*.
\end{aligned}$$

In the second equality we used the induction assumption and Lemma 3.6. □

Lemma 3.8. *Suppose that each h_n^* is unique. Let H be a predictable d-dimensional process such that*

$$\mathsf{E}\sum_{n=1}^{N}\alpha^n\langle H_n,\Delta S_n\rangle = R_0^* c,$$

$$u_0(\langle H,\Delta S\rangle)\ge -c.$$

Then $H=H^*$ a.s.

`Proof`. Fix $n\in\{1,\dots,N\}$. It follows from (3.8) that

$$\alpha\mathsf{E}(\langle H_n,X_n\rangle - R_n^* u_n(\langle H,X\rangle)\,|\,\mathcal{F}_{n-1})\le -R_{n-1}^* u_{n-1}(\langle H,X\rangle).$$

If this inequality is strict with a strictly positive probability, then, going backwards from n to 0 and using (3.9), we see that, for $m=n-1,\dots,0$, the inequality

$$\mathsf{E}\biggl[\sum_{k=m+1}^{N}\alpha^{k-m}\langle H_k,X_k\rangle\,\bigg|\,\mathcal{F}_m\biggr]\le -R_m^* u_m(\langle H,X\rangle)$$

is strict with a strictly positive probability. For $m=0$, we obtain a contradiction. Thus,

$$\alpha\mathsf{E}(\langle H_n,X_n\rangle - R_n^* u_n(\langle H,X\rangle)\,|\,\mathcal{F}_{n-1}) = -R_{n-1}^* u_{n-1}(\langle H,X\rangle).$$

If $\mathsf{P}(u_{n-1}(\langle H,X\rangle)=\infty)>0$, then, going backwards from $n-1$ to 0, we check that $\mathsf{P}(u_m(\langle H,X\rangle)=\infty)>0$ for $m=n-1,\dots,0$. For $m=0$, we obtain a contradiction. Thus, $u_{n-1}(\langle H,X\rangle)<\infty$. Combining this with the preceding conclusion, we get, for a.e. ω,

$$\alpha\mathsf{E}(\langle h,\widetilde{X}\rangle - R_m^*\widetilde{V}) = -R_{n-1}^* v<\infty, \tag{3.10}$$

where v, h, $\widetilde{X}$, and $\widetilde{V}$ are the same as in the proof of Lemma 3.5.

Let us prove that

$$h=|v|h_n^*, \tag{3.11}$$

$$\widetilde{V}=v(\langle h_n^*,\widetilde{V}\rangle - a(h_n^*))^+\quad \text{if } n<N. \tag{3.12}$$

If $v=0$, then it follows from equality (3.7), assumption (3.2), and Lemma 3.3 that $h=0$ and $\widetilde{V}=0$, so that (3.11) and (3.12) are trivially satisfied. Let now $v<0$. Denote $\overline{h}=h/|y|$, $\overline{V}=\widetilde{V}/|y|$. Then $\overline{V}\le 0$ due to Lemma 3.3 and $u_\mu(\langle\overline{h},\widetilde{X}\rangle+\overline{V})=-1$ due to (3.7). It follows from Lemma 3.2 applied to $X=\langle\overline{h},\widetilde{X}\rangle$ and $Y=\langle\overline{h},\widetilde{X}\rangle+\overline{V}$ that

$$R_{n-1}^*=\alpha\mathsf{E}(\langle\overline{h},\widetilde{X}\rangle - R_n^*\overline{V})\le\alpha\mathsf{E}(\langle\overline{h},\widetilde{X}\rangle + R_n^*(\langle\overline{h},\widetilde{X}\rangle - a(\overline{h})^+)\le R_{n-1}^*,$$

where the equality follows from (3.10). Both inequalities here should be equalities. As h_n^* is unique, we have $\overline{h}=h_n^*$, which proves (3.11). If $n<N$, then $R_n^*>0$, so that $\mathsf{E}\overline{V} = -\mathsf{E}(\langle \overline{h}, \widetilde{X}\rangle - a(\overline{h}))^+$ and, due to Lemma 3.2, $\overline{V} = -(\langle \overline{h}, \widetilde{X}\rangle - a(\overline{h}))^+$. This proves (3.12).

Thus, (3.11)–(3.12) are true for a.e. ω. This means that

$$H_m = -u_{m-1}(\langle H, X\rangle)h_m^*, \quad m = 1, \dots, N,$$
$$u_m(\langle H, X\rangle) = u_{m-1}(\langle H, X\rangle)(\langle h_m^*, X_m\rangle - a(h_m^*))^+, \quad m = 1, \dots, N-1.$$

Clearly, $u_0(\langle H, X\rangle) = -c$ since otherwise we can consider the process $H' = 2H \wedge Hc/u_0(\langle H, X\rangle)$, for which $u_0(\langle H', X\rangle) \geq -c$ and

$$\mathsf{E}\sum_{k=1}^{N} \alpha^k \langle H', X\rangle > \mathsf{E}\sum_{k=1}^{N} \alpha^k \langle H, X\rangle = R_0^* c,$$

which contradicts Lemma 3.5. Recalling the definition of H^*, we see that $H = H^*$. □

`Proof of Theorem 3.1.` Note that, for any $H \in \mathcal{A}$, $\langle H_n, \Delta S_n\rangle = \langle H_n', X_n\rangle$, where $H_n' = \sigma_n^t H_n$. Now, the fact that $R_0^* c$ is the optimal value and H^* is an optimal strategy follows from Lemmas 3.5 and 3.7. The uniqueness part follows from Lemma 3.8. □

4 Static Optimization Problem

Multidimensional case. The results of the previous section reduce the original dynamic problem (3.3) to the static one

$$\begin{cases} \mathsf{E}\big[\langle h, X\rangle + R(\langle h, X\rangle - a)^+\big] \longrightarrow \max, \\ h \in \mathbb{R}^d, \ a \in \mathbb{R}, \\ u_\mu(\langle h, X\rangle \wedge a) \geq -1, \end{cases} \tag{4.1}$$

where X is a d-dimensional random vector, $R \in \mathbb{R}_+$, μ is a positive measure on $(0,1]$ with $\mu((0,1]) \leq 1$, $\mu((0,1)) > 0$, and u_μ is given by (2.2). Indeed, it is clear that, for the optimal solution (h^*, a^*) of (4.1), we have $a^* = a(h^*)$, where $a(h)$ is defined by (3.4). Assumption (3.2) is transformed into: $-\infty < u_\mu(\langle h, X\rangle) < 0$ for any $h \in \mathbb{R}^d \setminus \{0\}$.

Note that (4.1) is not a concave maximization problem because the map $(h, a) \mapsto \mathsf{E}[\langle h, X\rangle + R(\langle h, X\rangle - a)^+]$ is convex.

A particularly important case of (3.1) is that of a conditionally Gaussian model, which corresponds to the case, when X is Gaussian. So, let us assume that X is Gaussian with mean $m \neq 0$ and a non-degenerate covariance matrix C. The theorem below shows that then (4.1) is equivalent to the one-dimensional problem

$$\begin{cases} \mathsf{E}\big[g\langle C^{-1}m, X\rangle + R(g\langle C^{-1}m, X\rangle - a)^+\big] \longrightarrow \max, \\ g \in \mathbb{R}_+, \ a \in \mathbb{R}, \\ u_\mu(g\langle C^{-1}m, X\rangle \wedge a) \geq -1. \end{cases} \tag{4.2}$$

Theorem 4.1. *A pair (h,a) is a solution of (4.1) if and only if $h=gC^{-1}m$, where (g,a) is a solution of (4.2).*

`Proof`. Denote $h_0=C^{-1}m$. We have

$$\operatorname*{argmax}_{h\in\mathbb{R}^d\setminus\{0\}}\frac{\mathsf{E}\langle h,X\rangle}{(\mathsf{Var}\langle h,X\rangle)^{1/2}}=\operatorname*{argmax}_{h\in\mathbb{R}^d\setminus\{0\}}\frac{\langle C^{1/2}h,C^{-1/2}m\rangle}{\langle C^{1/2}h,C^{1/2}h\rangle^{1/2}}=\{gh_0:g\in(0,\infty)\}.$$

Consequently, for any $h\notin\{gC^{-1}m:g\in(0,\infty)\}$, $h\neq 0$, we have

$$\mathsf{Law}\langle h,X\rangle=\mathsf{Law}(\gamma\langle h_0,X\rangle-\beta)$$

with some $\gamma>0$, $\beta>0$. This implies that

$$\begin{aligned}
&\max\bigl\{\mathsf{E}\bigl[g\langle h,X\rangle+R(g\langle h,X\rangle-a)^+\bigr]:g\in\mathbb{R}_+,\ a\in\mathbb{R},\ u_\mu(g\langle h,X\rangle\wedge a)\geq -1\bigr\}\\
&=\mathsf{E}\bigl[g^*\langle h,X\rangle+R(g^*\langle h,X\rangle-a^*)^+\bigr]\\
&<\mathsf{E}\bigl[g^*\gamma\langle h_0,X\rangle+R(g^*\gamma\langle h_0,X\rangle-a^*)^+\bigr]\\
&\leq\max\bigl\{\mathsf{E}\bigl[g\langle h_0,X\rangle+R(g\langle h_0,X\rangle-a)^+\bigr]:g\in\mathbb{R}_+,\ a\in\mathbb{R},\ u_\mu(g\langle h_0,X\rangle\wedge a)\geq -1\bigr\}.
\end{aligned}$$

The second inequality here follows from the line

$$u_\mu(g^*\gamma\langle h_0,X\rangle\wedge a^*)\geq u_\mu(g^*\langle h,X\rangle\wedge a^*)\geq -1.$$

This yields the desired statement. □

`Remarks`. (i) In the non-Gaussian case one can approach (4.1) numerically. For this, an efficient procedure for calculating u_μ is needed. One of equivalent representations of u_μ (see [20; Th. 4.64]), which is more convenient for numerical calculations than (2.2), is

$$u_\mu(\xi)=\int_{\mathbb{R}}xd\Psi_\mu(F_\xi(x)),$$

where

$$\Psi_\mu(x)=\int_0^x\int_{[y,1]}\lambda^{-1}\mu(d\lambda)dy$$

and F_ξ is the distribution function of ξ.

(ii) If $\mu=\delta_\lambda$ i.e. u_μ is Tail V@R, then the numerical procedure for solving (4.1) can further be simplified by the use of the method proposed by Rockafellar and Uryasev [28]. First, note that (4.1) is equivalent to the problem

$$\begin{cases}u_\mu(\langle h,X\rangle\wedge a)\longrightarrow\max,\\ h\in\mathbb{R}^d,\ a\in\mathbb{R},\\ \mathsf{E}\bigl[\langle h,X\rangle+R(\langle h,X\rangle-a)^+\bigr]=1.\end{cases}\tag{4.3}$$

According to the Rockafellar-Uryasev result, this problem is, in turn, equivalent to

$$\begin{cases} q - \lambda^{-1}\mathsf{E}(q - \langle h, X\rangle \wedge a)^+ \longrightarrow \max, \\ q \in \mathbb{R},\ h \in \mathbb{R}^d,\ a \in \mathbb{R}, \\ \mathsf{E}\big[\langle h, X\rangle + R(\langle h, X\rangle - a)^+\big] = 1 \end{cases} \tag{4.4}$$

in the sense that (h, a) is a solution of (4.3) if and only if there exists q such that (q, h, a) is a solution of (4.4). The advantage of (4.4) is that it does not contain expressions like u_λ but only has standard operations like taking the positive part.

One-dimensional case. Let us consider the one-dimensional version of (4.1). We assume that $\mathsf{E}X > 0$.

Lemma 4.2. *For $d = 1$, problem (4.1) is equivalent to*

$$\begin{cases} \mathsf{E}[hX + R(hX - a)^+] \longrightarrow \max, \\ h \in (0, \infty),\ a \in \mathbb{R}, \\ u_\mu(hX \wedge a) \ge -1 \end{cases} \tag{4.5}$$

in the sense that $h > 0$ for any solution (h, a) of (4.1).

`Proof`. For any $h < 0$ and $a \in \mathbb{R}$ such that $u_\mu(hX \wedge a) \ge -1$, we have

$$\begin{aligned} \mathsf{E}[hX + R(hX - a)^+] &\le R\mathsf{E}(hX - a)^+ \le -R\mathsf{E}(hX \wedge a) \\ &\le -Ru_\mu(hX \wedge a)/u_\mu((0,1]) \le R/\mu((0,1]). \end{aligned}$$

The third inequality follows from the property $\mathsf{E}\xi \ge u_\mu(\xi)/\mu((0,1])$, which is easily derived from the definition of u_μ.

For $h = 0$ and $a = -\mu((0,1])^{-1}$, we have $\mathsf{E}[hX + R(hX - a)^+] = R/\mu((0,1])$.

Furthermore,

$$\begin{aligned} &\frac{\partial}{\partial h}\Big|_{h=0, a=-\mu((0,1])^{-1}} \mathsf{E}[hX + R(hX - a)^+] > 0, \\ &\frac{\partial}{\partial h}\Big|_{h=0, a=-\mu((0,1])^{-1}} u_\mu(hX \wedge a) = 0. \end{aligned}$$

Hence,

$$\sup\{\mathsf{E}[hX + R(X - a)^+] : h \in (0, \infty),\ u_\mu(hX \wedge a) \ge -1\} > R/\mu((0,1]).$$

This completes the proof. □

Let us now study problem (4.5). Clearly, it is equivalent to the problem

$$\frac{\mathsf{E}[X + R(X - b)^+]}{-u_\mu(X \wedge b)} \longrightarrow \max, \quad b \in \mathbb{R} \tag{4.6}$$

in the sense that if b^* is a solution of (4.6), then $h^* = -u_\mu(X \wedge b^*)^{-1}$, $a^* = h^* b^*$ is a solution of (4.5); if (h^*, a^*) is a solution of (4.5), then $b^* = a^*/h^*$ is a solution of (4.6). Let us assume that X has a density that is strictly positive inside some interval and is zero outside this interval.

First consider the case $R = 0$. Then it is clear that the solution of (4.5) has the form

$$b^* = \inf\{x \in \mathbb{R} : u_\mu(X \wedge x) = u_\mu(X)\} = q_s,$$

where $s = \sup\{x : x \in \operatorname{supp} \mu\}$ and q_λ denotes the λ-quantile of X. So, in this case the optimal value in (4.6), which coincides with the optimal value in (4.5), is $-\mathsf{E}X/u_\mu(X)$. When constructing R_n^* through (3.5), we need to solve (4.6) iteratively, at each step plugging in as R the optimal value obtained at the previous step. As we have already described the solution for $R = 0$ (which is the starting value in the iterative procedure), we can now assume that $R \geq -\mathsf{E}X/u_\mu(X)$ (this assumption is needed in Theorem 4.3 below).

Introduce the notation

$$\Psi_\mu(x) = \int_0^x \int_y^1 \lambda^{-1} \mu(d\lambda) dy, \quad x \in (0,1]$$

and let D denote the distribution function of X. Denote $r = \sup\{x : x \in \operatorname{supp} \mathsf{Law}\, X\}$ and consider the function

$$G(b) = R \int_{\mathbb{R}} (x \wedge b)\, d(\Psi_\mu \circ D)(x) + \big[\mathsf{E}X + R\mathsf{E}(X-b)^+\big] \frac{1 - \Psi_\mu(D(b))}{1 - D(b)}, \quad x \in (-\infty, r). \quad (4.7)$$

Theorem 4.3. *The function G is decreasing, and there exists a number $b^* \in (-\infty, r)$ such that $G > 0$ on $(-\infty, b^*)$, $G(b^*) = 0$, and $G < 0$ on (b^*, r). The value b^* is the unique optimal solution of* (4.6).

Example 4.4. (i) Tail V@R. Let X be normal with mean m and variance σ^2; let $\mu = \delta_\lambda$, i.e. the risk measure is the dynamic Tail V@R. Then $\Psi_\mu = (\lambda^{-1} x) \wedge 1$ and

$$\begin{aligned} G(b) = {} & \frac{R}{\lambda\sqrt{2\pi}} \int_{-\infty}^{q_\lambda} (x \wedge b)\, e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx \\ & + \bigg(m + \frac{R\sigma}{\sqrt{2\pi}} \exp\Big\{-\frac{(m-b)^2}{2\sigma^2}\Big\} + R(m-b)\Phi\Big(\frac{m-b}{\sigma}\Big)\bigg) \frac{(\lambda - \Phi(b))^+}{\lambda \Phi(-b)}, \end{aligned}$$

where Φ is the cumulative distribution function of the standard normal distribution and q_λ denotes the λ-quantile of X.

(ii) Wang transform. Let X be the same as above and consider the case when $\Psi_\mu(x) = \Psi(x) = \Phi(\Phi^{-1}(x) + \theta)$, where Φ is the distribution function of the standard normal distribution and $\theta > 0$. The function Ψ is a concave increasing function $[0,1] \to [0,1]$, and the corresponding measure μ is given by $\mu(dx) = x\Psi''(x)dx$. This concave distortion was introduced by Wang [30]. We have

$$\Psi \circ D(x) = \Phi\Big(\Phi^{-1}\Big(\Phi\Big(\frac{x-m}{\sigma}\Big)\Big) + \theta\Big) = \Phi\Big(\frac{x-m}{\sigma} + \theta\Big) = \Phi\Big(\frac{x-m+\sigma\theta}{\sigma}\Big),$$

i.e. $\Psi \circ D$ is the distribution function of a Gaussian random variable with mean $m - \sigma\theta$ and variance σ^2. Then

$$\begin{aligned} G(b) &= R\int_{\mathbb{R}} (x \wedge b)\, q(x)\, dx + \Big(m + R\int_{\mathbb{R}} (x-b)^+ p(x)\, dx\Big)\frac{Q(X > b)}{P(X > b)} \\ &= Rm - R\sigma\theta - \frac{R\sigma}{\sqrt{2\pi}}\exp\Big\{-\frac{(m-\sigma\theta-b)^2}{2\sigma^2}\Big\} - (m-\sigma\theta-b)\Phi\Big(\frac{m-\sigma\theta-b}{\sigma}\Big) \\ &\quad + \Big(m + \frac{R\sigma}{\sqrt{2\pi}}\exp\Big\{-\frac{(m-b)^2}{2\sigma^2}\Big\} + R(m-b)\Phi\Big(-\frac{m-b}{\sigma}\Big)\Big)\frac{\Phi\big(\frac{m-\sigma\theta-b}{\sigma}\big)}{\Phi\big(\frac{m-b}{\sigma}\big)}, \end{aligned}$$

where P is the normal distribution with mean m and variance σ^2, Q is the normal distribution with mean $m - \sigma\theta$ and variance σ^2, p is the density of P, and q is the density of Q.

Lemma 4.5. *We have*

$$\lim_{b \to -\infty} G(b) = (1+R)\mathsf{E}X,$$
$$\lim_{b \uparrow r} G(b) = R\int_{\mathbb{R}} x d(\Psi_\mu \circ D)(x) + \mu(\{1\})\mathsf{E}X.$$

Furthermore, G *is decreasing on* $(-\infty, r)$ *and is strictly decreasing on* $\{x : G(x) > \lim_{b\uparrow r} G(b)\}$.

`Proof`. For $\lambda \in (0,1]$, denote $\Psi_\lambda(x) = (\lambda^{-1}x) \wedge 1$ (note that this is the function Ψ_μ corresponding to $\mu = \delta_\lambda$). Then

$$\Psi_\mu(x) = \int_{(0,1]} \Psi_\lambda(x)\mu(d\lambda), \quad x \in (0,1],$$

and hence,

$$G(b) = \int_{(0,1]} G_\lambda(b)\mu(d\lambda), \quad b \in (-\infty, r),$$

where G_λ is defined by (4.7) with Ψ_μ replaced by Ψ_λ. Therefore, it is sufficient to prove the lemma for the case, when $\mu = \delta_\lambda$. In this case G takes the form

$$G(b) = R\mathsf{E}(X \wedge b \,|\, X < q_\lambda) + [\mathsf{E}X + R\mathsf{E}(X-b)^+]\frac{\mathsf{P}(X > b \,|\, X < q_\lambda)}{\mathsf{P}(X > b)},$$

where q_λ is the λ-quantile of X.

For $\lambda = 1$, the statement is trivial. Let now $\lambda \in (0,1)$. The expressions for the limits of $G_\lambda(b)$ are clear (to prove the first of them, take into account the property

$\lim_{b\to-\infty} b\mathsf{P}(X<b)=0$, which follows from the integrability of X), and we should only check the monotonicity properties. The right-hand derivative of G_λ exists a.e. with respect to the Lebesgue measure on $(-\infty,r)$. A direct calculation shows that

$$(G_\lambda)'_+(b) = -(\mathsf{E}X + R\mathsf{E}(X-b)^+)p(b)\mathsf{P}(X>q_\lambda)\lambda^{-1}(\mathsf{P}(X>b))^{-2}<0, \quad b\in(-\infty,q_\lambda),$$

where p is the density of the distribution of X. Clearly, G_λ is constant on $[q_\lambda,r)$. This yields the desired monotonicity. □

`Proof of Theorem 4.3.` We have (see [20; Th. 4.64])

$$u_\mu(X\wedge b) = \int_{\mathbb{R}} (x\wedge b)\, d(\Psi_\mu\circ D)(x) \tag{4.8}$$

and so, for the function

$$F(b) = -\frac{\mathsf{E}X + R\mathsf{E}(X-b)^+}{u_\mu(X\wedge b)} = -\frac{\mathsf{E}X + R\mathsf{E}(X-b)^+}{\int_{\mathbb{R}}(x\wedge b)\, d(\Psi_\mu\circ D)(x)}, \quad b\in\mathbb{R},$$

we have

$$F'_+(b) = \frac{G(b)\,\mathsf{P}(X>b)}{\left(\int_{\mathbb{R}}(x\wedge b)\, d(\Psi_\mu\circ D)(x)\right)^2}, \quad b\in\mathbb{R}.$$

According to Lemma 4.5 and the assumption $R\ge -\mathsf{E}X/u_\mu(X)$, we have

$$\lim_{b\to-\infty} G(b) = (1+R)\mathsf{E}X>0,$$

$$\lim_{b\uparrow r} G(b) = R\int_{\mathbb{R}} x d(\Psi_\mu\circ D)(x) + \mu(\{1\})\mathsf{E}X \le -\mathsf{E}X + \mu(\{1\})\mathsf{E}X<0.$$

Obviously, G is continuous. Due to Lemma 4.5, there exists a unique root $b^*\in(-\infty,r)$ of the equation $G(b)=0$ and $G>0$ on $(-\infty,b^*)$, $G<0$ on (b^*,r). Thus, b^* is the unique optimal value in (4.6). □

5 Asymptotic Behavior

Let X be a random variable such that $\mathsf{E}X>0$, $-\infty<u_\mu(X)<0$, X has a density that is strictly positive inside some interval and is zero outside this interval. Define the sequences (R_n) and (h_n) by: $R_0=0$,

$$R_n = \max_{h\in\mathcal{H}} \alpha\mathsf{E}[hX + R_{n-1}(hX-a(h))^+], \quad n=1,2,\dots,$$

$$h_n = \operatorname*{argmax}_{h\in\mathcal{H}} \mathsf{E}[hX + R_{n-1}(hX-a(h))^+], \quad n=1,2,\dots.$$

Note that the values R_n^* of Section 3 are related to R_n by the equality $R_n^* = R_{N-n}$. Denote $a_n = a(h_n)$, where $a(h)$ is defined by (3.4). We will now study the asymptotic behavior of R_n, h_n, and a_n as $n\to\infty$. This corresponds to letting $N\to\infty$ in (3.3).

Introduce the notation

$$\varphi(R)=\max_{h\in\mathcal{H}}\alpha\mathsf{E}[hX+R(hX-a(h))^+],\quad R\in\mathbb{R}_+$$

and let β be the optimal value in the problem

$$\begin{cases}\mathsf{E}(hX-a)^+\longrightarrow\max,\\ h\in(0,\infty),\ a\in\mathbb{R},\\ u_\mu(hX\wedge a)\ge -1.\end{cases}\tag{5.1}$$

Obviously, this problem is equivalent to

$$\frac{\mathsf{E}(X-b)^+}{-u_\mu(X\wedge b)}\longrightarrow\max,\quad b\in\mathbb{R}.$$

The same arguments as those in the previous section show that the solution b^* of the latter problem is the unique root of the equation

$$\int_{\mathbb{R}}(x\wedge b)\,d(\Psi_\mu\circ D)(x)+\frac{1-\Psi_\mu(D(b))}{1-D(b)}\int_{\mathbb{R}}(x-b)^+dD(x)=0,\quad b\in(-\infty,r),$$

where Ψ_μ and r were introduced in the previous section. Thus, the solution (h^*,a^*) of (5.1) has the form $h^*=-u_\mu(X\wedge b^*)^{-1}$, $a^*=h^*b^*$.

Theorem 5.1. **(i)** *The values R_n are increasing in n. The values h_n are determined uniquely and are decreasing in n. The values a_n are decreasing in n.*

(ii) *If $\beta<1$, then $R_\infty=\lim_n R_n$ is finite and is the unique root of the equation $\varphi(R)=R$. Furthermore, the pair $(h_\infty,a_\infty)=\lim_n(h_n,a_n)$ is the unique solution of problem* (4.5) *with $R=R_\infty$.*

(iii) *If $\beta\ge 1$, then $\lim_n R_n=\infty$. Furthermore, the pair $(h_\infty,a_\infty)=\lim_n(h_n,a_n)$ is the unique solution of problem* (5.1).

`Proof.` **(i)** The property that R_n increase in n is clear. Note that $R_1=-\mathsf{E}X/u_\mu(X)$. According to Theorem 4.3, the value

$$b(R)=\operatorname*{argmax}_{b\in\mathbb{R}}\frac{\mathsf{E}[X+R(X-b)^+]}{-u_\mu(X\wedge b)},\quad R\in[-\mathsf{E}X/u_\mu(X),\infty)$$

is the unique root of the equation $G(R,b)=0$, where $G(R,b)$ is the function defined in Theorem 4.3. We have

$$\begin{aligned}\frac{\partial}{\partial R}G(R,b)&=\int_{\mathbb{R}}(x\wedge b)\,d(\Psi_\mu\circ D)(x)+\mathsf{E}(X-b)^+\frac{1-\Psi_\mu(D(b))}{1-D(b)}\\&=R^{-1}G(R,b)-R^{-1}\mathsf{E}X\frac{1-\Psi_\mu(D(b))}{1-D(b)}.\end{aligned}$$

For $b = b(R)$, this partial derivative is strictly negative. As G is decreasing in R, we see that $b(R)$ is decreasing in R.

In view of (4.8), we have for $h > 0$,

$$\frac{\partial}{\partial h} u_\mu(hX \wedge a) = \frac{\partial}{\partial h} \int_{\mathbb{R}} (hx \wedge a)\, d(\Psi_\mu \circ D)(x) = \int_{\mathbb{R}} xI(x < a/h)\, d(\Psi_\mu \circ D)(x) \le 0,$$

$$\frac{\partial}{\partial a} u_\mu(hX \wedge a) = \frac{\partial}{\partial a} \int_{\mathbb{R}} I(x > a/h)\, d(\Psi_\mu \circ D)(x) = 1 - \Psi_\mu(D(a/h)) \ge 0.$$

At each point (h, a) at least one of these inequalities is strict. Therefore, the function $a(h)$ is increasing on $\mathbb{R}_+$.

For any h_1, h_2, a_1, a_2, such that $u_\mu(h_1 X \wedge a_1) \ge -1$ and $u_\mu(h_2 X \wedge a_2) \ge -1$, we have

$$\begin{aligned} &u_\mu(\theta h_1 X + (1-\theta) h_2 X) \wedge (\theta a_1 + (1-\theta) a_2)) \\ &\ge u_\mu(\theta(h_1 X \wedge a_1) + (1-\theta)(h_2 X \wedge a_2)) \\ &\ge \theta u_\mu(h_1 X \wedge a_1) + (1-\theta) u_\mu(h_2 X \wedge a_2) \ge -1, \quad \theta \in [0,1], \end{aligned}$$

so that the function $a(h)$ is convex.

According to Lemma 4.2, the pair (h_n, a_n) is found as the intersection of the ray $\{(h, b(R_{n-1})h) : h \in (0, \infty)\}$ with the graph of the function $a(h)$. Obviously, h_n is unique. Using the properties of $a(h)$ established above and keeping in mind the monotonicity of R_n in n and the monotonicity of $b(R)$ in R, we see that h_n and a_n decrease in n.

(ii) We have $\varphi(0) = -\alpha \mathsf{E} X / u_\mu(X) > 0$. Furthermore, due to the compactness of $\mathcal{H}$, $\varphi(R)/R \to \beta$ as $R \to \infty$. In particular, $\varphi(R) < R$ for large R. Moreover, φ is continuous. Now, it follows from the recurrent relation $R_n = \varphi(R_{n-1})$ that $R_n \to R_\infty$, where $R_\infty < \infty$ is the smallest root of the equation $\varphi(R) = R$. The function $\varphi(R)$ is convex as the maximum of linear functions. As $\varphi(0) > 0$ and $\varphi(R) < R$ for large R, the equation $\varphi(R) = R$ has a unique root.

Fix $h \in \mathcal{H}$ and $a \in \mathbb{R}$ such that $u_\mu(hX \wedge a) \ge -1$. Then

$$\mathsf{E}[h_n X + R_{n-1}(h_n X - a_n)^+] \ge \mathsf{E}[hX + R_{n-1}(hX - a)^+], \quad n \in \mathbb{N}.$$

Passing on to the limit, we see that

$$\mathsf{E}[h_\infty X + R_\infty(h_\infty X - a_\infty)^+] \ge \mathsf{E}[hX + R_\infty(hX - a)^+].$$

As $u_\mu(h_n X \wedge a_n) \ge -1$, we get $u_\mu(h_\infty X \wedge a_\infty) \ge -1$. Thus, (h_∞, a_∞) is a solution of (4.5) with $R = R_\infty$. The uniqueness of a solution follows from Lemma 4.2 and Theorem 4.3.

(iii) According to (i), there exists $R_\infty = \lim_n R_n$. Suppose that $R_\infty < \infty$. Then $\varphi(R_\infty) = R_\infty$. Note that the maximum in (5.1) is attained due to the compactness of the set $\{h : u_\mu(\langle h, X\rangle) \ge -1\}$. Similar arguments as in Lemma 4.2 show that the optimal solution (h^*, a^*) of (5.1) satisfies $h^* > 0$. Then

$$\varphi(R_\infty) \ge \alpha \mathsf{E}[h^* X + R_\infty(h^* X - a^*)^+] > \alpha R_\infty \mathsf{E}(h^* X - a^*)^+ = \beta R_\infty \ge R_\infty.$$

The obtained contradiction shows that $R_\infty = \infty$.

Fix $h \in \mathcal{H}$ and $a \in \mathbb{R}$ such that $u_\mu(hX \wedge a) \ge -1$. Then

$$\mathsf{E}[R_{n-1}^{-1} h_n X + (h_n X - a_n)^+] \ge \mathsf{E}[R_{n-1}^{-1} hX + (hX - a)^+], \quad n \in \mathbb{N}.$$

Passing on to the limit and taking into account the compactness of $\mathcal{H}$, we see that

$$\mathsf{E}(h_\infty X - a_\infty)^+ \ge \mathsf{E}(hX - a)^+,$$

i.e. (h_∞, a_∞) is a solution of (5.1). □

Denote by $H^*(N)$ the optimal strategy in (3.3). The values h_n^* of Section 3 are related to the values h_n by the equality $h_n^* = h_{N-n}$. So,

$$H_n^*(N) = C_{n-1}^*(N)\sigma_n^{-1} h_{N-n+1},$$
$$C_n^*(N) = C_{n-1}^*(N)(h_{N-n+1} X_n - a(h_{N-n+1}))^+.$$

Proceeding by the induction in n, we conclude that $H_n^*(N) \xrightarrow[N\to\infty]{} H_n^*$ for each $n = 1, 2, \dots$, where H_n^* and C_n^* are defined by: $C_0^* = c$,

$$H_n^* = C_{n-1}^* \sigma_n^{-1} h_\infty, \quad n = 1, 2, \dots, \tag{5.2}$$
$$C_n^* = C_{n-1}^*(h_\infty X_n - a(h_\infty))^+, \quad n = 1, 2, \dots. \tag{5.3}$$

In particular, if we denote by $\tau(N)$ the stopping time $\inf\{n : H_n^* = 0\}$ (see the remark following Theorem 3.1), then $\tau(N) \to \tau$, where $\tau = \inf\{n : h_\infty X_n \le a(h_\infty)\}$ has a geometric distribution with the parameter $\mathsf{P}(h_\infty X > a_\infty)$. Note in particular that τ does not depend on the capital constraint c. So, the optimal strategy for the model with the infinite time horizon has the feature: the trading is switched off after some random time, which is finite a.s.

Conclusion

We have considered the problem of reward maximization with a risk constraint, the risk being measured by a discrete-time dynamic coherent risk measure. The class of models under consideration includes all the multidimensional conditionally Gaussian models.

In Section 3, we reduce the original problem to a sequence of static problems, so that the optimal solution is constructed by first finding the auxiliary values h_n^* and R_n^* proceeding backwards from N to 0 and then finding the processes C_n^* and H_n^* going forwards from 1 to N (Theorem 3.1).

The obtained static problem is studied in Section 4. We first reduce the multidimensional problem to the dimension one in the case of conditionally Gaussian model (Theorem 4.1). Then we propose an explicit solution of the corresponding one-dimensional

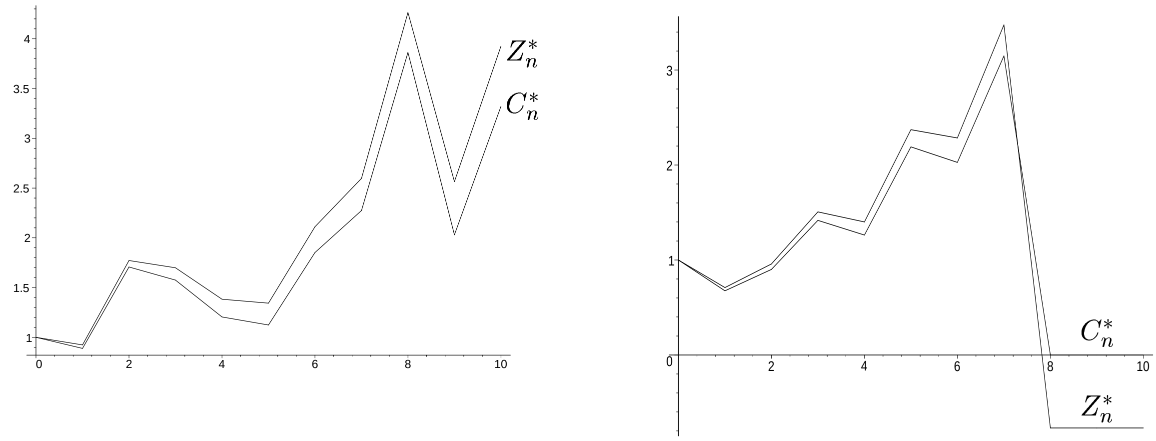

Figure 1. Simulated paths of the process C_n^* given by (5.3) and the cumulative capital process $Z_n^* = \sum_{k=1}^n \langle H_k^*, \Delta S_k \rangle$, where H_n^* is given by (5.2). We have taken the one-dimensional conditionally Gaussian model with $\mathsf{E} X_n = 0.3$, $\mathsf{Var}\, X_n = 1$ (in practice, this corresponds to the case, when the unit time period is one year; then the return/volatility ratio for the equity is about 0.3), $\alpha = 0.8$, and have traced the evolution of Z_n^*, C_n^* for 10 steps. In this example, $R_\infty = 1.04$, $h_\infty = 0.48$, $a_\infty = -0.97$. The parameter $\mathsf{P}(h_\infty X > a_\infty)$ equals 0.99, so that the probability that τ occurs within the first 10 steps is $1 - 0.99^{10} = 0.1$. The graph on the left illustrates the situation, when τ has not occurred within 10 steps; the graph on the right illustrates the (rare) event, when τ has occurred within 10 steps.

problem, which is expressed through the (unique) root of some explicitly given function (Theorem 4.3).

Finally, we analyze in Section 5 the behaviour of the solution as the time horizon tends to infinity. It is shown (Theorem 5.1) that the optimal strategy and the optimal values tend monotonically to some limits, which are expressed through the roots of some explicitly given functions.

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