PRICING WITH COHERENT RISK

A.S. Cherny

Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Probability Theory,
119992 Moscow, Russia.
E-mail: alexander.cherny@gmail.com
Webpage: http://mech.math.msu.su/~cherny

Abstract. This is the first of two papers dealing with applications of coherent risk measures to basic problems of financial mathematics. In this paper, we study applications to pricing in incomplete markets. We prove the fundamental theorem of asset pricing for the No Good Deals pricing technique based on coherent risks. The model considered includes static and dynamic models as well as models with infinitely many assets, and models with transaction costs. In particular, we prove that in a dynamic model with proportional transaction costs the fair price interval converges to the fair price interval in the frictionless model as the coefficient of transaction costs tends to zero.

Moreover, we study some problems in the "pure" theory of risk measures. In particular, we introduce the notion of a generator that opens the way for geometric constructions. Based on this notion, we give a simple geometric solution of the capital allocation problem.

Key words and phrases: Capital allocation, coherent risk measure, extreme measure, generator, No Good Deals, RAROC, risk contribution, risk-neutral measure, support function, Tail V@R, transaction costs, Weighted V@R.

1 Introduction

1. Overview. The three basic pillars of finance are:

- optimal investment;
- pricing and hedging;
- risk measurement and management.

The most well-known financial theories related to the first pillar are the Markowitz mean-variance analysis and Sharpe's CAPM, which are often termed "the first revolution in finance". The most well-known result related to the second pillar is the Black-Scholes-Merton formula, which is often termed "the second revolution in finance". Recently, a very important innovation has appeared in connection with the third pillar. In 1997, Artzner, Delbaen, Eber and Heath [4], [5] introduced the concept of a *coherent risk measure* as a new way of measuring risk. Since 1997, the theory of coherent risk measures has rapidly been evolving and is already termed in some sources "the third revolution in finance" (see [68]). Let us mention, in particular, the papers [1], [3], [21], [34], [35], [43], [49], [69] and the reviews [22], [36; Ch. 4], [60]. Currently, one of the major tasks is the problem of proper risk measurement in the dynamic setting; see, in particular, [12], [27], [41], [55], and [57].

The theory of coherent risk measures is important not only for risk measurement. Indeed, risk ($\approx$ uncertainty) is at the very basis of the whole finance, and therefore, a new way of looking at risk yields new approaches to other problems of finance, in particular, to those related to the first and the second pillars. Nowadays, more and more research is aimed at *applications* of coherent risk measures to various problems of finance. In fact, the whole finance can be built based on coherent risks.

One of the major goals of modern financial mathematics is providing adequate price bounds for derivative contracts in incomplete markets. It is known that No Arbitrage price bounds in incomplete markets are typically unacceptably wide, and fundamentally new ideas are required to narrow these bounds. Recently, a promising approach to this problem termed *No Good Deals* (*NGD*) pricing has been proposed in [6], [17]. Let us illustrate its idea by an example. Consider a contract that with probability 1/2 yields nothing and with probability 1/2 yields 1000 USD. The No Arbitrage (NA) price interval for this contract is $(0, 1000)$. But if the price of the contract is, for instance, 15 USD, then everyone would be willing to buy it, and the demand would not match the supply. Thus, 15 USD is an unrealistic price because it yields a good deal, i.e. a trade that is attractive to most market participants. The technique of the NGD pricing is based on the assumption that good deals do not exist.

A problem that arises immediately is how to define a good deal. In [6] and [17] it is understood as a trade with unusually high profit/loss ratio. Here Cochrane and Saá-Requejo [17] employed Sharpe ratio, while Bernardo and Ledoit [6] used another profit/loss ratio. Černý and Hodges [11] proposed a generalization of both definitions (see also the paper [7] by Bjork and Slinko, which extends the results of [17]).

Alternatively the technique of the NGD pricing can be motivated as follows. When a trader sells a contract, he/she would charge for it a price, with which he/she would be able to superreplicate the contract. In theory the superreplication is typically understood almost surely, but in practice an agent looks for an offsetting position such that the risk of his/her overall portfolio would stay within the limits prescribed by his/her management (the almost sure superreplication is virtually impossible in practice). These considerations lead to the NGD pricing with a good deal defined as a trade with negative risk (this definition of a good deal is alternative to the definitions in the papers [6], [7], [11], [17], where a good deal is defined as a trade with high profit/loss ratio), while hedging means minimization of risk.

Traditional measures of risk are variance and V@R (Value at Risk)[1]. If risk is measured as variance, then its minimization corresponds to the variance-optimal hedging introduced by Duffie and Richardson [29] and developed in the papers [53], [61], [62] (see also [36; Ch. 10]). If risk is measured by V@R, then its minimization corresponds to the quantile hedging introduced by Föllmer and Leukert [33] (see also [36; Ch. 8]).

But it is better to measure risk through coherent risk measures rather than as variance or V@R. The corresponding pricing technique has already been considered in several papers. Carr, Geman, and Madan [9] (see also the review paper [10]) studied this technique in a probabilistic framework (although they do not use the term "good deal"), while Jaschke and Küchler [40] studied this technique in a topological space framework in the spirit of Harrison and Kreps [37] (see also the paper [66] by Staum, which extends the

[1] Let us recall that, for a portfolio earning a P&L X (P&L means the Profit&Loss, i.e. the difference between the terminal wealth and the initial wealth), Value at Risk is defined as $\mathrm{V@R}_\lambda(X) = -q_\lambda(X)$, where q_λ denotes the λ-quantile and $\lambda \in (0,1)$ is a fixed parameter (typically $\lambda = 0.05$ or $\lambda = 0.01$).

results of [40]). Let us also mention the papers [41], [50], and [57] dealing with the NGD pricing based on dynamic coherent risk measures and on convex ones.

2. Goal of the paper. This is the first of a series of papers dealing with applications of coherent risk measures to the basic problems of finance (the other paper in the series is [15]). In this paper, we study applications to pricing in incomplete markets. We study the NGD technique in two forms: in the first one a good deal is understood as a trade with negative risk, while in the second one a good deal is understood as a trade with unusually high profit/risk ratio (it turns out that the second technique is reduced to the first one simply by changing the risk measure). A popular topic in financial mathematics is obtaining various forms of the fundamental theorem of asset pricing for the No Arbitrage pricing technique (it will suffice to mention the papers [14], [20], [23], [24], [37], [38], [39], [44], [46], [56], [58], [67], [70]; see also [64; Ch. 5,7]). We prove this theorem for the NGD pricing based on coherent risks (Theorem 3.4). The obtained theorem is rather general[2]: it is applicable to arbitrary probability spaces, a wide class of coherent risk measures (including all the natural risk measures), continuous-time dynamic models, models with infinitely many assets, and models with transaction costs[3]. As a corollary of the obtained theorem, we get the form of the fair price interval (Corollary 3.6).

A problem that has attracted attention in several papers is as follows. Consider a model with proportional transaction costs. Is it true that the upper (resp., lower) price of a contingent claim in this model tends to the upper (resp., lower) price of this claim in the frictionless model as the coefficient of transaction costs tends to zero? It was shown in [14], [19], [51], and [65] that, for NA prices, the answer to this question is negative already in the Black-Scholes-Merton model (the contingent claim considered in these papers is a European call option). This result might be interpreted as follows: the NA technique is useless in continuous-time models with transaction costs. In this paper (Theorem 3.18), we prove that, for NGD prices, the answer to the above question is positive. This is done within a framework of a general model (the price follows an arbitrary process) with an infinite number of assets and an arbitrary contingent claim (satisfying only some integrability condition). The advantage of the NGD pricing is not only that this result is true in such a general setting, but also that its proof is short.

Although this series of papers deals primarily with applications of coherent risk measures to problems of finance, we also establish some results and give several definitions related to "pure" risk measures (these are needed for applications). In particular, we introduce the notion of an *extreme measure*. The results of this paper and [15] show that this notion is very convenient and important; it appears in the outcomes of several pricing techniques proposed in [15] and in considerations of the equilibrium problem in [15]. In the present paper, we provide a solution of the *capital allocation* problem in terms of extreme measures (Theorem 2.12). Let us remark that this problem was considered in [22], [26], [31], [48], [54], and [69].

Parallel with the measurement of outstanding risks, a very important problem is measuring the risk contribution of a subportfolio to a "big" portfolio. Based on our solution

[2]One can say that our approach generalizes [9], but this is a generalization in many directions: the paper [9] assumes an unrealistic world of a finite Ω and a finite set of probabilistic scenarios determining a coherent risk measure (all the natural risk measures are defined through an infinite set of scenarios; see Subsection 2.1). Moreover, [9] considers only a static model with a finite number of assets and without transaction costs. Good deals defined through profit/loss ratio are not considered in [9].

[3]The No Arbitrage fundamental theorem of asset pricing for models with transaction costs was studied, in particular, in the papers [14], [25], [42], [45], [47], [59].

of the capital allocation problem, we propose several equivalent definitions of the *coherent risk contribution*.

Another notion we introduce is the notion of a *generator*. It establishes a bridge between coherent risks and convex analysis, opening the way for geometry. In particular, we provide (see Figure 1) a geometric solution of the capital allocation problem (thus there are two solutions: a probabilistic one is given in terms of extreme measures, while a geometric one is given in terms of generators). We also provide a geometric solution of the pricing and hedging problem (Figure 4) for a model with a finite number of assets. Furthermore, we provide in [15] geometric solutions of several optimization problems, optimality pricing problems, and the equilibrium problem. In fact, for most problems considered in this series of papers, we provide two sorts of results:

- a geometric result applicable to a model with a finite number of assets is given in terms of generators;
- a probabilistic result applicable to a general model is typically given in terms of extreme measures.

To sum up, let us mention several advantages of the NGD pricing based on coherent risks as compared to techniques, like arbitrage pricing, variance-optimal hedging, quantile hedging, and NGD techniques based on other profit/risk ratios:

- The obtained fair price intervals are smaller than the No Arbitrage price intervals. In particular, there is convergence of price intervals in models with transaction costs.
- The technique is flexible in the sense that it can be applied with various risk measures. The other methods do not have this flexibility.
- The technique admits a form, where a good deal is understood as a trade with negative risk, as well as a form, where a good deal is defined through the profit/risk ratio. Moreover, the second technique is reduced to the first one simply by changing the risk measure. The other methods do not have this stability property.
- As a coherent risk measure is defined through a family of probability measures, a rich duality theory arises.
- Introducing the notion of a generator establishes a relationship with geometry and opens the way for applications of convex analysis.

3. Structure of the paper. Section 2 deals with "pure" risk measures rather than with their applications. Subsection 2.1 recalls some basic definitions related to coherent risks. In Subsection 2.2, we introduce the L^1-spaces associated with a coherent risk measure (these are employed in the technical conditions in theorems below). Subsection 2.3 presents the definition of an extreme measure. In Subsection 2.4, we provide a solution of the capital allocation problem. Subsection 2.5 deals with equivalent definitions of risk contribution.

Section 3 is related to the NGD pricing. In Subsections 3.1 and 3.2, we study two forms of this technique for a general model. In Subsections 3.3–3.5, we consider some particular cases of this model: a static model with a finite number of assets (for which fair price intervals admit a simple geometric description; see Figure 3), a continuous-time dynamic model, and a continuous-time dynamic model with transaction costs. Furthermore, in Subsection 3.6, we provide a geometric solution of the hedging problem for a static model with a finite number of assets.

2 Coherent Risk Measures

2.1 Basic Definitions

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. The following definition was introduced in [4], [5]. These papers considered only a finite Ω, and in that case the continuity axiom (e) (see below) is not needed. It was added for a general Ω by Delbaen [21]. In theory, it is more convenient to deal with negatives of coherent risk measures termed coherent utility functions (this notion was introduced in [22]). This enables one to get rid of numerous minus signs.

Definition 2.1. A *coherent utility function on* L^∞ is a map $u : L^\infty \to \mathbb{R}$ with the properties:

(a) (Superadditivity) $u(X+Y) \geq u(X) + u(Y)$;
(b) (Monotonicity) If $X \leq Y$, then $u(X) \leq u(Y)$;
(c) (Positive homogeneity) $u(\lambda X) = \lambda u(X)$ for $\lambda \in \mathbb{R}_+$;
(d) (Translation invariance) $u(X+m) = u(X) + m$ for $m \in \mathbb{R}$;
(e) (Fatou property) If $|X_n| \leq 1$, $X_n \xrightarrow{\mathsf{P}} X$, then $u(X) \geq \limsup_n u(X_n)$.

The corresponding *coherent risk measure* is $\rho(X) = -u(X)$.

Remark. Typically, a coherent risk measure is defined only via conditions (a)–(d), and then one has coherent risk measures with the Fatou property. However, only such risk measures are useful, and for this reason, we find it more convenient to add (e) as a basic axiom.

The theorem below was established in [5] for the case of a finite Ω (in this case the axiom (e) is not needed) and in [21] for the general case. We denote by $\mathcal{P}$ the set of probability measures on $\mathcal{F}$ that are absolutely continuous with respect to P. Throughout the paper, we identify measures from $\mathcal{P}$ (these are typically denoted by Q) with their densities with respect to P (these are typically denoted by Z).

Theorem 2.2 (Basic representation theorem). *A function u satisfies conditions* (a)–(e) *if and only if there exists a non-empty set $\mathcal{D} \subseteq \mathcal{P}$ such that*

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X, \quad X \in L^\infty. \tag{2.1}$$

Remark. Measures from $\mathcal{D}$ are sometimes called the *probabilistic scenarios* defining ρ.

So far, coherent risk measures have been defined on bounded random variables. On the other hand, most distributions used in financial mathematics are unbounded (like the normal or the lognormal ones). It seems hopeless to axiomatize coherent risk measures on the space L^0 of all random variables and then to obtain a representation theorem. Instead, we take representation (2.1) as the basis and extend it to L^0.

Definition 2.3. A *coherent utility function on* L^0 is a map $u : L^0 \to [-\infty, \infty]$ defined as

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0, \tag{2.2}$$

where $\mathcal{D} \subseteq \mathcal{P}$ and $\mathsf{E}_\mathsf{Q} X$ is understood as $\mathsf{E}_\mathsf{Q} X^+ - \mathsf{E}_\mathsf{Q} X^-$ with the convention $\infty - \infty = -\infty$. The corresponding *coherent risk measure* is $\rho(X) = -u(X)$.

Clearly, a set $\mathcal{D}$, for which representations (2.1) and (2.2) are true, is not unique. However, there exists the largest such set given by $\{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X \geq u(X) \text{ for any } X\}$. We introduce the following definition.

Definition 2.4. Let u be a coherent utility function on L^∞ (resp., L^0). We will call the largest set, for which (2.1) (resp., (2.2)) is true, the *determining set of* u.

`Remark.` Clearly, the determining set is convex. For coherent utility functions on L^∞, it is also L^1-closed. However, for coherent utility functions on L^0, it is not necessarily L^1-closed. As an example, take a positive unbounded random variable X_0 such that $\mathsf{P}(X_0 = 0) > 0$ and consider $\mathcal{D}_0 = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X_0 = 1\}$. Clearly, the determining set $\mathcal{D}$ of the coherent utility function $u(X) = \inf_{\mathsf{Q} \in \mathcal{D}_0} \mathsf{E}_\mathsf{Q} X$ satisfies $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X_0 \geq 1\}$. On the other hand, the L^1-closure of $\mathcal{D}_0$ contains measures concentrated on $\{X_0 = 0\}$.

`Important Remark.` Let $\mathcal{D}$ be an L^1-closed convex subset of $\mathcal{P}$. (Let us note that a particularly important case is where $\mathcal{D}$ is L^1-closed, convex, and uniformly integrable; this condition will be needed in a number of places below). Define a coherent utility function u by (2.2). Then $\mathcal{D}$ is the determining set of u. Indeed, assume that the determining set $\widetilde{\mathcal{D}}$ is greater than $\mathcal{D}$, i.e. there exists $\mathsf{Q}_0 \in \widetilde{\mathcal{D}} \setminus \mathcal{D}$. Then, by the Hahn-Banach theorem, we can find $X_0 \in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0} X_0 < \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X$, which is a contradiction. The same argument shows that $\mathcal{D}$ is also the determining set of the restriction of u to L^∞.

In what follows, we will always consider coherent utility functions on L^0.

Example 2.5. (i) *Tail V@R* (the terms *Average V@R*, *Conditional V@R*, and *Expected Shortfall* are also used) is the risk measure corresponding to the coherent utility function

$$u_\lambda(X) = \inf_{\mathsf{Q} \in \mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X,$$

where $\lambda \in (0, 1]$ and

$$\mathcal{D}_\lambda = \left\{\mathsf{Q} \in \mathcal{P} : \frac{d\mathsf{Q}}{d\mathsf{P}} \leq \lambda^{-1}\right\}. \tag{2.3}$$

Suppose that X has a continuous distribution and set $Z_* = \lambda^{-1} I(X \leq q_\lambda)$, where q_λ is the λ-quantile of X. Then

$$\begin{aligned}(X - q_\lambda)Z - (X - q_\lambda)Z^* = (X - q_\lambda)(Z - \lambda^{-1})I(X - q_\lambda < 0)\\ + (X - q_\lambda)ZI(X - q_\lambda > 0) \geq 0\end{aligned} \tag{2.4}$$

for any $Z \in \mathcal{D}_\lambda$, which implies that

$$u_\lambda(X) = \mathsf{E} Z_* X = \mathsf{E}(X \,|\, X \leq q_\lambda)$$

(this motivates the term Tail V@R). For more information on this risk measure, see [3], [21; Sect. 6], [22; Sect. 7], [36; Sect. 4.4], [60; Sect. 1.3].

(ii) *Weighted V@R on* L^∞ (the term *spectral risk measure* is also used) is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \int_{(0,1]} u_\lambda(X)\mu(d\lambda), \quad X \in L^\infty, \tag{2.5}$$

where μ is a probability measure on $(0, 1]$.

Weighted V@R on L^0 is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \inf_{\mathsf{Q}\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0,$$

where $\mathcal{D}_\mu$ is the determining set of u_μ on L^∞ (equality (2.5) remains valid on L^0; see [16; Th. 3.2]).

Let us remark that, under some regularity conditions on μ, Weighted V@R possesses some nice properties that are not shared by Tail V@R. In a sense, it is "smoother" than Tail V@R. We consider Weighted V@R as one of the most important classes of coherent risk measures. For a detailed study of this risk measure, see [1], [2], [28], [49] as well as the paper [16], which is in some sense the continuation of the present paper and [15]. □

2.2 Spaces L^1_w and L^1_s

For a subset $\mathcal{D}$ of $\mathcal{P}$, we introduce the *weak* and *strong* L^1*-spaces*

$$L^1_w(\mathcal{D}) = \{X \in L^0 : u(X) > -\infty,\, u(-X) > -\infty\},$$
$$L^1_s(\mathcal{D}) = \Big\{X \in L^0 : \lim_{n\to\infty} \sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q}|X| I(|X|>n) = 0\Big\}.$$

Clearly, $L^1_s(\mathcal{D}) \subseteq L^1_w(\mathcal{D})$. If $\mathcal{D} = \{\mathsf{Q}\}$ is a singleton, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D}) = L^1(\mathsf{Q})$, which motivates the notation.

In general, $L^1_s(\mathcal{D})$ might be strictly smaller than $L^1_w(\mathcal{D})$. Indeed, let X_0 be a positive unbounded random variable with $\mathsf{P}(X_0 = 0) > 0$ and let $\mathcal{D} = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X_0 = 1\}$. Then $X_0 \in L^1_w(\mathcal{D})$, but $X_0 \notin L^1_s(\mathcal{D})$. (One can also construct a similar counterexample with an L^1-closed set $\mathcal{D}$; see Example 2.11.) However, as shown by the proposition below, in most natural situations weak and strong L^1-spaces coincide.

Proposition 2.6. (i) *If $\mathcal{D}_\lambda$ is the determining set of Tail V@R (see Example 2.5 (i)), then $L^1_w(\mathcal{D}_\lambda) = L^1_s(\mathcal{D}_\lambda)$.*

(ii) *If $\mathcal{D}_\mu$ is the determining set of Weighted V@R (see Example 2.5 (ii)), then $L^1_w(\mathcal{D}_\mu) = L^1_s(\mathcal{D}_\mu)$.*

(iii) *If all the densities from $\mathcal{D}$ are bounded by a single constant and $\mathsf{P} \in \mathcal{D}$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.*

(iv) *If $\mathcal{D}$ is a convex combination $\sum_{n=1}^N a_n \mathcal{D}_n$, where $\mathcal{D}_1, \dots, \mathcal{D}_N$ are such that $L^1_w(\mathcal{D}_n) = L^1_s(\mathcal{D}_n)$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.*

(v) *If $\mathcal{D} = \operatorname{conv}(\mathcal{D}_1, \dots, \mathcal{D}_N)$, where $\mathcal{D}_1, \dots, \mathcal{D}_N$ are such that $L^1_w(\mathcal{D}_n) = L^1_s(\mathcal{D}_n)$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.*

Lemma 2.7. *If μ is a convex combination $\sum_{n=1}^\infty a_n \delta_{\lambda_n}$, then the determining set $\mathcal{D}_\mu$ of Weighted V@R corresponding to μ has the form $\sum_{n=1}^\infty a_n \mathcal{D}_{\lambda_n}$, where $\mathcal{D}_\lambda$ is given by (2.3).*

`Proof`. Denote $\sum_n a_n \mathcal{D}_{\lambda_n}$ by $\mathcal{D}$. Clearly, $\mathcal{D}$ is convex. Fix $X \in L^\infty$. It is easy to see that, for any n, the minimum of expectations of $\mathsf{E}XZ$ over $Z \in \mathcal{D}_{\lambda_n}$ is attained on the random variable $Z_k = \lambda_k^{-1} I(X < q_{\lambda_k}) + c_k I(X = q_{\lambda_k})$, where q_{λ_k} is the λ_k-quantile of X (the proof is similar to the reasoning of Example 2.5 (i)). Hence, the minimum of expectations $\mathsf{E}_\mathsf{P} XZ$ over $Z \in \mathcal{D}$ is attained. By the James theorem (see [32]), $\mathcal{D}$ is weakly compact. In particular, it is L^1-closed.

Obviously, $u_\mu(X) = \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X$ for any $X \in L^\infty$. Taking into account the Important Remark following Definition 2.4, we get $\mathcal{D}_\mu = \mathcal{D}$. □

`Proof of Proposition 2.6.` The only nontrivial statement is (ii). In order to prove it, consider the measures $\widetilde{\mu} = \sum_{k=1}^\infty a_k \delta_{2^{-k}}$, $\bar{\mu} = \sum_{k=1}^\infty a_k \delta_{2^{-k+1}}$, where $a_k = \mu((2^{-k}, 2^{-k+1}])$. As $u_{\widetilde{\mu}} \le u_\mu \le u_{\bar{\mu}}$, we have $\mathcal{D}_{\widetilde{\mu}} \supseteq \mathcal{D}_\mu \supseteq \mathcal{D}_{\bar{\mu}}$. By Lemma 2.7,

$$\mathcal{D}_{\widetilde{\mu}} = \left\{ \sum_{k=1}^\infty a_k Z_k : Z_k \in \mathcal{D}_{2^{-k}} \right\}, \qquad \mathcal{D}_{\bar{\mu}} = \left\{ \sum_{k=1}^\infty a_k Z_k : Z_k \in \mathcal{D}_{2^{-k+1}} \right\}.$$

Take $X \in L^1_w(\mathcal{D}_\mu)$. Consider $Z_k = 2^{k-1} I(X < q_k) + c_k I(X = q_k)$, where q_k is the 2^{-k+1}-quantile of X and c_k is chosen in such a way that $\mathsf{E}_\mathsf{P} Z_k = 1$. Then

$$\mathsf{E}_\mathsf{P} Z_k X = \min_{Z\in\mathcal{D}_{2^{-k+1}}} \mathsf{E}_\mathsf{P} ZX.$$

The density $Z_0 = \sum_{k=1}^\infty a_k Z_k$ belongs to $\mathcal{D}_{\bar{\mu}}$ and

$$\mathsf{E}_\mathsf{P} Z_0 X = \min_{Z\in\mathcal{D}_{\bar{\mu}}} \mathsf{E}_\mathsf{P} ZX.$$

In view of the inclusion $X \in L^1_w(\mathcal{D}_\mu) \subseteq L^1_w(\mathcal{D}_{\bar{\mu}})$, the latter quantity is finite. Thus,

$$\sum_{k=1}^\infty a_k \min_{Z\in\mathcal{D}_{2^{-k+1}}} \mathsf{E}_\mathsf{P} ZX > -\infty,$$

which implies that

$$\sum_{k=1}^\infty a_k \min_{Z\in\mathcal{D}_{2^{-k+1}}} \mathsf{E}_\mathsf{P} Z(-X^-) > -\infty.$$

The same estimate is true for X^+, and therefore,

$$\sum_{k=1}^\infty a_k \sup_{Z\in\mathcal{D}_{2^{-k}}} \mathsf{E}_\mathsf{P} Z|X| \le 2\sum_{k=1}^\infty a_k \sup_{Z\in\mathcal{D}_{2^{-k+1}}} \mathsf{E}_\mathsf{P} Z|X| < \infty. \tag{2.6}$$

It is clear that $X \in L^1$, and thus, for each k,

$$\sup_{Z\in\mathcal{D}_{2^{-k}}} \mathsf{E}_\mathsf{P} Z|X|I(|X| > n) \le 2^k \mathsf{E}_\mathsf{P} |X| I(|X| > n) \xrightarrow[n\to\infty]{} 0.$$

This, combined with (2.6), yields

$$\begin{aligned}\sup_{Z\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{P} Z|X|I(|X| > n) &\le \sup_{Z\in\mathcal{D}_{\widetilde{\mu}}} \mathsf{E}_\mathsf{P} Z|X|I(|X| > n)\\ &= \sum_{k=1}^\infty a_k \sup_{Z\in\mathcal{D}_{2^{-k}}} \mathsf{E}_\mathsf{P} Z|X|I(|X| > n) \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

2.3 Extreme Measures

Definition 2.8. Let u be a coherent utility function with the determining set $\mathcal{D}$. Let $X \in L^0$. We will call a measure $\mathsf{Q} \in \mathcal{D}$ an *extreme measure* for X if $\mathsf{E}_\mathsf{Q} X = u(X) \in (-\infty, \infty)$. The set of extreme measures will be denoted by $\mathcal{X}_\mathcal{D}(X)$.

Let us recall some general facts related to the weak topology on L^1. The *weak topology* on L^1 is induced by the duality between L^1 and L^∞ and is usually denoted as $\sigma(L^1, L^\infty)$. The Dunford-Pettis criterion states that a set $\mathcal{D} \subseteq \mathcal{P}$ is weakly compact if and only if it is weakly closed and uniformly integrable. Furthermore, an application of the Hahn-Banach theorem shows that a convex set $\mathcal{D} \subseteq \mathcal{P}$ is weakly closed if and only if it is L^1-closed.

Proposition 2.9. *If the determining set $\mathcal{D}$ is weakly compact and $X \in L^1_s(\mathcal{D})$, then $\mathcal{X}_\mathcal{D}(X) \neq \emptyset$.*

Proof. It is clear that $u(X) \in (-\infty, \infty)$. Find a sequence $Z_n \in \mathcal{D}$ such that $\mathsf{E}_\mathsf{P} Z_n X \to u(X)$. This sequence has a weak limit point $Z_\infty \in \mathcal{D}$. Clearly, the map $\mathcal{D} \ni Z \mapsto \mathsf{E}_\mathsf{P} ZX$ is weakly continuous. Hence, $\mathsf{E}_\mathsf{P} Z_\infty X = u(X)$, which means that $Z_\infty \in \mathcal{X}_\mathcal{D}(X)$. □

Example 2.10. (i) If u corresponds to Tail V@R of order λ (see Example 2.5 (i)) and X has a continuous distribution, then it is easy to see from (2.4) that $\mathcal{X}_\mathcal{D}(X)$ consists of a unique density $\lambda^{-1} I(X \le q_\lambda)$, where q_λ is a λ-quantile of X.

(ii) If u corresponds to Weighted V@R with a weighting measure μ (see Example 2.5 (ii)) and X has a continuous distribution, then $\mathcal{X}_\mathcal{D}(X)$ consists of a unique density $g(X)$, where $g(x) = \int_{[F(x),1]} \lambda^{-1} \mu(d\lambda)$ and F is the distribution function of X (see [16; Sect. 6]). Note that this density reflects the risk aversion of an agent possessing a portfolio that produces the P&L (Profit&Loss) X. □

The condition that $\mathcal{D}$ should be weakly compact is very mild and is satisfied for the determining sets of most natural coherent risk measures. Obviously, the determining set $\mathcal{D}_\lambda$ of Tail V@R is weakly compact. The determining set $\mathcal{D}_\mu$ of Weighted V@R is weakly compact; this follows from the explicit representation of this set provided in [8] (the proof can also be found in [36; Th. 4.73] or [60; Th. 1.53]); this can also be seen from the representation of $\mathcal{D}_\mu$ provided in [16; Th. 4.6].

The following example shows that the condition $X \in L^1_s(\mathcal{D})$ in Proposition 2.8 cannot be replaced by the condition $X \in L^1_w(\mathcal{D})$.

Example 2.11. Let $\Omega = [0,1]$ be endowed with the Lebesgue measure. Consider $Z_n = \sqrt{n} I_{[0,1/n]} + 1 - 1/\sqrt{n}$, $n \in \mathbb{N}$. Then $Y_n := Z_n - 1 \xrightarrow{L^1} 0$, and therefore, the set

$$\mathcal{D} = \left\{ 1 + \sum_{n=1}^\infty a_n Y_n : a_n \ge 0,\ \sum_{n=1}^\infty a_n \le 1 \right\}$$

is convex, L^1-closed, and uniformly integrable. Thus, $\mathcal{D}$ is weakly compact. Now, consider $X(\omega) = -1/\sqrt{\omega}$. Then $\mathsf{E}_\mathsf{P} Z_n X = -4 + 2/\sqrt{n}$. Thus, $\inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X = -4$, while there exists no $\mathsf{Q} \in \mathcal{D}$ such that $\mathsf{E}_\mathsf{Q} X = -4$. □

2.4 Capital Allocation

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the determining set $\mathcal{D}$, and let $X^1, \dots, X^d \in L^1_w(\mathcal{D})$ be the discounted P&Ls produced by different components of a firm (P&L means the Profit&Loss, i.e. the difference between the terminal wealth and the initial wealth, so that it takes on both positive and negative values). We will use the notation $X = (X^1, \dots, X^d)$.

Informally, the capital allocation problem is the following. How is the total risk $\rho\big(\sum_i X^i\big)$ being split between the components $1,\dots,d$? In other words, we are looking for a vector $(x^1,\dots,x^d)$ such that x^i means the part of the risk carried by the i-th component. Taking $x^i=\rho(X^i)$ does not work because $\sum_i\rho(X^i)\neq\rho\big(\sum_i X^i\big)$. The following definition of a capital allocation is taken from [22; Sect. 9]. In fact, it is closely connected with the coalitional games (see [26]).

Problem (capital allocation): Find $x^1,\dots,x^d\in\mathbb{R}$ such that

$$\sum_{i=1}^d x^i = u\Big(\sum_{i=1}^d X^i\Big), \tag{2.7}$$

$$\forall h^1,\dots,h^d\in\mathbb{R}_+,\quad \sum_{i=1}^d h^i x^i \ge u\Big(\sum_{i=1}^d h^i X^i\Big). \tag{2.8}$$

We will call a solution of this problem a *utility allocation between* $X^1,\dots,X^d$. A *capital allocation* is defined as a utility allocation with the minus sign.

From the financial point of view, $-x^i$ is the contribution of the i-th component to the total risk of the firm, or, equivalently, the capital that should be allocated to this component. In order to illustrate the meaning of (2.8), consider the example $h^i=I(i\in J)$, where J is a subset of $\{1,\dots,d\}$. Then (2.8) transforms into the condition $\sum_{i\in J}(-x^i)\le\rho\big(\sum_{i\in J}X^i\big)$, which means that the capital allocated to a part of the firm does not exceed the risk carried by that part ((2.8) states the same for "weighted" parts of the firm).

Let us introduce the notation $G=\mathrm{cl}\{\mathsf{E}_\mathsf{Q}X:\mathsf{Q}\in\mathcal{D}\}$, where "cl" denotes the closure. Note that G is convex and compact. We will call it the *generating set* or simply the *generator* for X and u. This term is justified by the line

$$u(\langle h,X\rangle)=\inf_{\mathsf{Q}\in\mathcal{D}}\mathsf{E}_\mathsf{Q}\langle h,X\rangle=\inf_{\mathsf{Q}\in\mathcal{D}}\langle h,\mathsf{E}_\mathsf{Q}X\rangle=\min_{x\in G}\langle h,x\rangle,\quad h\in\mathbb{R}^d. \tag{2.9}$$

Note that the last expression is a classical object of convex analysis known as the *support function* of the convex set G.

Theorem 2.12. *The set U of utility allocations between $X^1,\dots,X^d$ has the form*

$$U=\operatorname*{argmin}_{x\in G}\langle e,x\rangle, \tag{2.10}$$

where $e=(1,\dots,1)$. Furthermore, for any utility allocation x, we have

$$\forall h^1,\dots,h^d\in\mathbb{R},\quad \sum_{i=1}^d h^i x^i \ge u\Big(\sum_{i=1}^d h^i X^i\Big). \tag{2.11}$$

If moreover $X^1,\dots,X^d\in L_s^1(\mathcal{D})$ and $\mathcal{D}$ is weakly compact, then

$$U=\Big\{\mathsf{E}_\mathsf{Q}X:\mathsf{Q}\in\mathcal{X}_\mathcal{D}\Big(\sum_{i=1}^d X^i\Big)\Big\}. \tag{2.12}$$

Proof. (The proof is illustrated by Figure 1.) For $h \in \mathbb{R}^d$, we set

$$L(h) = \Big\{ x \in \mathbb{R}^d : \langle h, x\rangle = \min_{y \in G} \langle h, y\rangle \Big\},$$
$$M(h) = \Big\{ x \in \mathbb{R}^d : \langle h, x\rangle \geq \min_{y \in G} \langle h, y\rangle \Big\}.$$

It is seen from (2.9) that the set of points $x \in \mathbb{R}^d$ that satisfy (2.7) is $L(e)$. The set of points x that satisfy (2.8) is $\bigcap_{h \in \mathbb{R}^d_+} M(h) = G + \mathbb{R}^d_+$. The set of points x that satisfy (2.11) is $\bigcap_{h \in \mathbb{R}^d} M(h) = G$. This proves (2.10) and (2.11). Furthermore, the set $\{\mathsf{E}_\mathsf{Q} X : \mathsf{Q} \in \mathcal{D}\}$ is closed (the proof is similar to the proof of Proposition 2.9). Now, equality (2.12) follows immediately from (2.10) and the definition of $\mathcal{X}_\mathcal{D}$. □

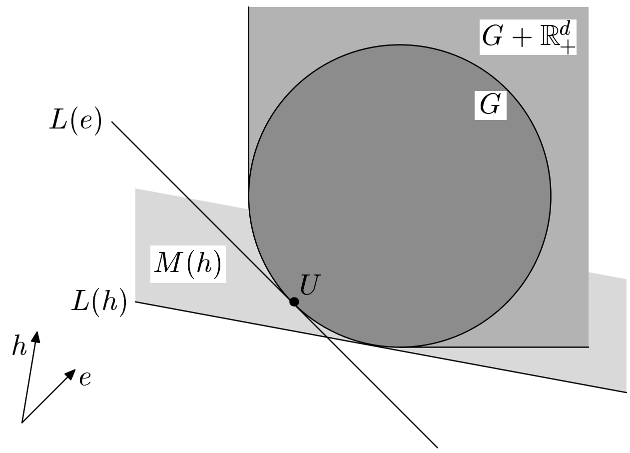

Figure 1. Solution of the capital allocation problem

If G is strictly convex (i.e. its interior is non-empty and its border contains no interval), then a utility allocation is unique. However, in general it is not unique as shown by the example below.

Example 2.13. Let $d = 2$ and $X^2 = -X^1$. Then G is the interval with the endpoints $(u(X^1), -u(X^1))$ and $(-u(-X^1), u(-X^1))$. In this example, $U = G$. □

Let us now find the solution of the capital allocation problem in the Gaussian case.

Example 2.14. Let X have Gaussian distribution with mean a and covariance matrix C. Let u be a *law invariant* coherent utility function, i.e. $u(X)$ depends only on the distribution of X (note that Tail V@R and Weighted V@R satisfy this condition); we also assume that u is finite on Gaussian random variables.

Then there exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. Let L denote the image of $\mathbb{R}^d$ under the map $x \mapsto Cx$. Then the inverse $C^{-1} : L \to L$ is correctly defined. It is easy to see that

$$G = a + \{C^{1/2}x : \|x\| \leq \gamma\} = a + \{y \in L : \langle y, C^{-1}y\rangle \leq \gamma^2\}.$$

Let $e = (1, \dots, 1)$ and assume first that $Ce \neq 0$. In this case the utility allocation x_0 between $X^1, \dots, X^d$ is determined uniquely. In order to find it, note that, for any $y \in L$ such that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \langle x_0 - a + \varepsilon y, C^{-1}(x_0 - a + \varepsilon y)\rangle = 0,$$

we have $\langle e, y\rangle = 0$. This implies that $C^{-1}(x_0 - a) = \alpha \operatorname{pr}_L e$ with some constant α (pr_L denotes the orthogonal projection on L). Thus, $x_0 = a + \alpha Ce$. As x_0 should belong to the relative border of G (i.e. the border in the relative topology of $a + L$), we have $\langle x_0 - a, C^{-1}(x_0 - a)\rangle = \gamma^2$, i.e. $\alpha = -\gamma\langle e, Ce\rangle^{-1/2}$. As a result, the utility allocation between $X^1, \dots, X^d$ is $a - \gamma\langle e, Ce\rangle^{-1/2} Ce$.

Assume now that $Ce = 0$. This means that e is orthogonal to L, and then the set of utility allocations between $X^1, \dots, X^d$ is G.

Let us remark that in this example the solution of the capital allocation problem depends on u rather weakly, i.e. it depends only on γ. □

2.5 Risk Contribution

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the determining set $\mathcal{D}$, and $Y \in L^0$ be the discounted P&L produced by the whole firm.

Such a firm assesses the risk of any trade producing a P&L X not as $\rho(X)$, but rather as $\rho(W + X) - \rho(X)$. Below we define a risk contribution $\rho^c(X; W)$ in such a way that $\rho(\,\cdot\,; W)$ is a coherent risk measure and $\rho^c(X; W) \approx \rho(W + X) - \rho(W)$ provided that X is small as compared to W (the precise statement is Theorem 2.16).

Definition 2.15. The *utility contribution of X to Y* is

$$u^c(X; W) = \inf_{\mathsf{Q} \in \mathcal{X}_{\mathcal{D}}(Y)} \mathsf{E}_{\mathsf{Q}} X.$$

The *risk contribution of X to Y* is defined as $\rho^c(X; Y) = -u^c(X; Y)$.

The utility contribution is a coherent utility function provided that $\mathcal{X}_{\mathcal{D}}(Y) \neq \emptyset$.

If $\mathcal{D}$ is weakly compact and $X, Y \in L^1_s(\mathcal{D})$, then, by Theorem 2.12,

$$u^c(X; Y) = \inf\{x^1 : (x^1, x^2) \text{ is a utility allocation between } X, Y - X\}.$$

Using this formula, we can define risk contribution under the following conditions: $\mathcal{D}$ is arbitrary and $X, Y \in L^1_w(\mathcal{D})$.

If $\mathcal{D}$ is weakly compact, $X^1, \dots, X^d \in L^1_s(\mathcal{D})$, and $\mathcal{X}_{\mathcal{D}}\big(\sum_i X^i\big)$ is a singleton, then (in view of Theorem 2.12) the utility allocation between $X^1, \dots, X^d$ is unique and has the form

$$\bigg(u^c\Big(X^1; \sum_{i=1}^d X^i\Big), \dots, u^c\Big(X^d; \sum_{i=1}^d X^i\Big)\bigg).$$

This shows the relevance of the given definition. Another argument supporting this definition is the statement below.

Theorem 2.16. *If $\mathcal{D}$ is weakly compact and $X, Y \in L^1_s(\mathcal{D})$, then*

$$u^c(X; Y) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(u(Y + \varepsilon X) - u(Y)).$$

`Proof.` (The proof is illustrated by Figure 2.) Consider the generator $G = \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}}(X, Y) : \mathsf{Q} \in \mathcal{D}\}$ and set $b = \min\{y : (x, y) \in G\}$, $a = \min\{x : (x, b) \in G\}$. Note that $u^c(X; Y) = a$. The minimum $\min_{(x,y) \in G}\langle(\varepsilon, 1), (x, y)\rangle$ is attained at a point

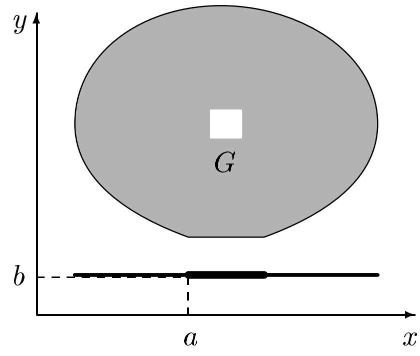

Figure 2

$(a(\varepsilon), b(\varepsilon))$. We have $a(\varepsilon) \le a$, $b(\varepsilon) \ge b$, and $(a(\varepsilon), b(\varepsilon)) \xrightarrow[\varepsilon\downarrow 0]{} (a, b)$. Furthermore, $\varepsilon a(\varepsilon) + b(\varepsilon) \le \varepsilon a + b$, which implies that $0 \le b(\varepsilon) - b \le \varepsilon(a - a(\varepsilon))$. As a result,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(u(Y + \varepsilon X) - u(Y)) &= \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(\varepsilon a(\varepsilon) + b(\varepsilon) - b)\\ &= a + \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(b(\varepsilon) - b) = a = u^c(X; Y).\end{aligned}$$

Example 2.17. **(i)** Let Y be a constant. In this case $\mathcal{X}_{\mathcal{D}}(Y) = \mathcal{D}$, so that $u^c(X; Y) = u(X)$.

(ii) Let $X = \alpha Y$ with $\alpha \in \mathbb{R}_+$. Then $u^c(X; Y) = \alpha u(Y)$.

(iii) Let X, Y have a jointly Gaussian distribution with mean $(\mathsf{E}X, \mathsf{E}Y)$ and covariance matrix C. Let u be a law invariant coherent utility function that is finite on Gaussian random variables. Then there exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. Assume that X and Y are not degenerate and $\mathsf{corr}(X, Y) \neq \pm 1$. It follows from Example 2.14 that

$$\begin{aligned}u^c(X; Y) &= \mathsf{E}X - \gamma\langle e_2, Ce_2\rangle^{-1/2} Ce_2\\ &= \mathsf{E}X - \gamma\,\frac{\mathsf{cov}(X, Y)}{(\mathsf{var}\, Y)^{1/2}}\\ &= \mathsf{E}X + (u(X) - \mathsf{E}X)\,\mathsf{corr}(X, Y),\end{aligned}$$

where $e_2 = (0, 1)$. In particular, if $\mathsf{E}X = \mathsf{E}Y = 0$, then

$$\frac{u^c(X; Y)}{u(X)} = \mathsf{corr}(X, Y) = \frac{\mathrm{V@R}^c(X; Y)}{\mathrm{V@R}(X)},$$

where $\mathrm{V@R}^c$ denotes the V@R contribution (for the definition, see [52; Sect. 7]). □

3 Good Deals Pricing

3.1 Utility-Based Good Deals Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, and A be a convex subset of L^0. From the financial point of view, A is the set of various discounted P&Ls that can be obtained in the model under consideration by employing various trading strategies (examples are given in Subsections 3.3–3.5). It will be called the *set of attainable P&Ls.* We will assume that A

is $\mathcal{D}$-consistent (see Definition 3.2 below). It is shown in Subsections 3.3–3.5 that this assumption is automatically satisfied for natural models.

First, we give the definition of a risk-neutral measure. Of course, this notion is a classical object of financial mathematics, but the particular definition we need is taken from [13] (it is adapted to the L^0-case).

Definition 3.1. A *risk-neutral measure* is a measure $\mathsf{Q} \in \mathcal{P}$ such that $\mathsf{E}_{\mathsf{Q}} X \le 0$ for any $X \in A$ (we use the convention $\mathsf{E}X = \mathsf{E}X^+ - \mathsf{E}X^-$, $\infty - \infty = -\infty$). The set of risk-neutral measures will be denoted by $\mathcal{R}$ or by $\mathcal{R}(A)$ if there is a risk of ambiguity.

Definition 3.2. We will say that A *is* $\mathcal{D}$*-consistent* if there exists a set $A' \subseteq A \cap L_s^1(\mathcal{D})$ such that $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A')$.

Definition 3.3. A model satisfies the *utility-based NGD* condition if there exists no $X \in A$ such that $u(X) > 0$.

Theorem 3.4 (Fundamental theorem of asset pricing). *A model satisfies the NGD condition if and only if* $\mathcal{D} \cap \mathcal{R} \neq \emptyset$.

Proof. The "if" part is obvious. Let us prove the "only if" part. Fix $X_1, \dots, X_M \in A'$. It follows from the weak continuity of the maps $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_{\mathsf{Q}} X_m$ that the set $G = \{\mathsf{E}_{\mathsf{Q}}(X_1, \dots, X_M) : \mathsf{Q} \in \mathcal{D}\}$ is compact. Clearly, G is convex. Suppose that $G \cap (-\infty, 0]^M = \emptyset$. Then there exist $h \in \mathbb{R}^M$ and $\varepsilon > 0$ such that $\langle h, x\rangle \ge \varepsilon$ for any $x \in G$ and $\langle h, x\rangle \le 0$ for any $x \in (-\infty, 0]^M$. Hence, $h \in \mathbb{R}_+^M$. Without loss of generality, $\sum_m h_m = 1$. Then $X = \sum_m h_m X_m \in A$ and $\mathsf{E}_{\mathsf{Q}} X \ge \varepsilon$ for any $\mathsf{Q} \in \mathcal{D}$, so that $u(X) > 0$. The obtained contradiction shows that, for any $X_1, \dots, X_M \in A'$, the set

$$B(X_1, \dots, X_M) = \{\mathsf{Q} \in \mathcal{D} : \mathsf{E}_{\mathsf{Q}} X_m \le 0 \text{ for any } m = 1, \dots, M\}$$

is non-empty. As $X_m \in L_s^1(\mathcal{D})$, the map $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_{\mathsf{Q}} X_m$ is weakly continuous, and therefore, $B(X_1, \dots, X_M)$ is weakly closed. Furthermore, any finite intersection of sets of this form is non-empty. As $\mathcal{D}$ is weakly compact, there exists a measure Q that belongs to each B of this form. Then $\mathsf{E}_{\mathsf{Q}} X \le 0$ for any $X \in A'$, which means that $\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}(A')$. As A is $\mathcal{D}$-consistent, $\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}$. □

Remarks. (i) As opposed to the fundamental theorems of asset pricing dealing with the NA condition and its strengthenings (see [13], [23], [24]), here we need not take any closure of A when defining the NGD. Essentially, this is the compactness of $\mathcal{D}$ that yields the fundamental theorem of asset pricing.

(ii) If $\mathcal{D} = \mathcal{P}$, then the NGD condition means that there exists no $X \in A$ with $\operatorname{essinf}_\omega X(\omega) > 0$. This is very close to the NA condition. However, in this case $\mathcal{D}$ is not uniformly integrable and Theorem 3.4 might be violated. Indeed, let $A = \{hX : h \in \mathbb{R}\}$, where X has uniform distribution on $[0,1]$. Then the NGD is satisfied, while $\mathcal{R} = \emptyset$.

Now, let $F \in L^0$ be the discounted payoff of a contingent claim.

Definition 3.5. A *utility-based NGD price* of F is a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGD condition. The set of the NGD prices will be denoted by $I_{NGD}(F)$.

Corollary 3.6 (Fair price interval). *For* $F \in L_s^1(\mathcal{D})$,

$$I_{NGD}(F) = \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}.$$

`Proof`. Denote $\{h(F-x) : h \in \mathbb{R}\}$ by $A(x)$. Clearly, $A + A(x)$ is $\mathcal{D}$-consistent (in order to prove this, it is sufficient to consider $A' + A(x)$). It follows from Theorem 3.4 that $x \in I_{NGD}(F)$ if and only if $\mathcal{D} \cap \mathcal{R}(A + A(x)) \neq \emptyset$. It is easy to check that $\mathsf{Q} \in \mathcal{R}(A + A(x))$ if and only if $\mathsf{Q} \in \mathcal{R}$ and $\mathsf{E}_\mathsf{Q} F = x$. This completes the proof. □

`Remark`. As opposed to the NA price intervals, the NGD price intervals are closed (this follows from the weak continuity of the map $\mathcal{D} \cap \mathcal{R} \mapsto \mathsf{E}_\mathsf{Q} F$).

To conclude the subsection, we will discuss the origin of $\mathcal{D}$. First of all, $\mathcal{D}$ might be the determining set of a coherent utility function like Tail V@R or Weighted V@R. The set $\mathcal{D}$ might also correspond to a weighted average or the minimum of several coherent utility functions. It is also possible that $\mathcal{D}$ originates from the classical utility maximization as described by the example below.

Example 3.7. Let $\mathsf{P}_1, \dots, \mathsf{P}_N$ be a family of probability measures, $U_1, \dots, U_N$ be a family of classical utility functions (i.e. smooth concave increasing functions $\mathbb{R} \to \mathbb{R}$), and $W_1, \dots, W_N$ be a family of random variables. From the financial point of view, P_n, U_n, and W_n are the subjective probability, the utility function, and the terminal wealth of the n-th market participant, respectively. Consider the measure $\mathsf{Q}_n = c_n U_n'(W_n)\mathsf{P}_n$, where c_n is the normalizing constant. Then, for any trading opportunity $X \in L^0$, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} U_n(W_n + \varepsilon X) = \mathsf{E}_{\mathsf{P}_n} U_n'(W_n) X = \mathsf{E}_{\mathsf{Q}_n} c_n^{-1} X \tag{3.1}$$

(we assume that all the expectations exist and integration is interchangeable with differentiation). Thus, an opportunity εX with a small $\varepsilon > 0$ is attractive to the n-th participant if and only if $\mathsf{E}_{\mathsf{Q}_n} X > 0$, so that Q_n might be called the valuation measure of the n-th participant. Take $\mathcal{D} = \operatorname{conv}(\mathsf{Q}_1, \dots, \mathsf{Q}_N)$ and consider the corresponding coherent utility function u. Then $u(X) > 0$ if and only if $\mathsf{E}_{\mathsf{Q}_n} X > 0$ for any n. In view of (3.1), this means that εX with some $\varepsilon > 0$ is attractive to any market participant (this is similar to the notion of a strictly acceptable opportunity introduced in [9]). Thus, in this example the NGD means the absence of a trading opportunity that is attractive to every agent. □

3.2 RAROC-Based Good Deals Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{RD} \subset \mathcal{P}$ be a convex weakly compact set, $\mathcal{PD}$ be an L^1-closed convex subset of $\mathcal{RD}$, and A be a $\mathcal{RD}$-consistent convex subset of L^0. We will call $\mathcal{PD}$ the *profit-determining set*. Thus, the profit of a position that yields a P&L X is $\inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E}_\mathsf{Q} X$. We will call $\mathcal{RD}$ the *risk-determining set*, so that the risk of a position that yields a P&L X is $-\inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X$. A canonical example is: $\mathcal{PD} = \{\mathsf{P}\}$ and $\mathcal{RD}$ is the determining set of a coherent utility function. The fact that $\mathcal{PD}$ need not be a singleton accounts for the ambiguity of the historical probability measure, or, in other words, for model risk. The financial interpretation of A is the same as above. Finally, we fix a positive number R meaning the upper limit on a possible RAROC.

Definition 3.8. The *Risk-Adjusted Return on Capital* (RAROC) for $X \in L^0$ is defined as

$$\mathrm{RAROC}(X) = \begin{cases} +\infty & \text{if } \inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E}_\mathsf{Q} X > 0 \text{ and } \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_\mathsf{Q} X \geq 0, \\ \dfrac{\inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E}_\mathsf{Q} X}{-\inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_\mathsf{Q} X} & \text{otherwise} \end{cases}$$

with the convention $\frac{0}{0} = 0$, $\frac{\infty}{\infty} = 0$.

Definition 3.9. A model satisfies the *RAROC-based NGD* condition if there exists no $X \in A$ such that $\mathrm{RAROC}(X) > R$.

Corollary 3.10 (Fundamental theorem of asset pricing). *A model satisfies the NGD condition if and only if*

$$\Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R} \neq \emptyset. \tag{3.2}$$

`Proof`. Let us first consider the case $R > 0$. Then, for any $X \in L^0$,

$$\mathrm{RAROC}(X) > R \iff \inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E}_{\mathsf{Q}}X + R \inf_{\mathsf{Q}\in\mathcal{RD}} \mathsf{E}_{\mathsf{Q}}X > 0 \iff \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}X > 0,$$

where $\mathcal{D} = \big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\big)$. Clearly, $\mathcal{D}$ is weakly compact and A is $\mathcal{D}$-consistent (note that $L_s^1(\mathcal{D}) = L_s^1(\mathcal{RD})$). Now, the statement follows from Theorem 3.4.

Let us now consider the case $R = 0$. Then the "if" part is obvious, and we should check the "only if" part. Take $A' \subseteq A \cap L_s^1(\mathcal{RD})$ such that $\mathcal{RD} \cap \mathcal{R} = \mathcal{RD} \cap \mathcal{R}(A')$. For any $X \in \operatorname{conv} A'$, $\inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E}_{\mathsf{Q}}X \le 0$. Repeating the arguments from the proof of Theorem 3.4, we get $\mathcal{PD} \cap \mathcal{R} \neq \emptyset$. □

Definition 3.11. A *RAROC-based NGD price* of a contingent claim F is a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{PD}, \mathcal{RD}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGD condition. The set of the NGD prices will be denoted by $I_{NGD}(F)$.

Corollary 3.12 (Fair price interval). *For $F \in L_s^1(\mathcal{D})$,*

$$I_{NGD}(F) = \Big\{\mathsf{E}_{\mathsf{Q}}F : \mathsf{Q} \in \Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R}\Big\}.$$

This statement follows from Corollary 3.10.

Let us compare the utility-based NGD and the RAROC-based NGD by a practical example.

Example 3.13. Let $\mathcal{PD} = \{\mathsf{P}\}$, $\mathcal{D}$ be the determining set of Tail V@R of order 0.05, $A = 0$ (i.e. there is no market), $R = 0.1$ (this a typical value of RAROC for a one-year period), and F be a Gaussian random variable with mean 0 and variance 1. Then an agent using the utility-based NGD would charge for it the price

$$\sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}F = -u_{0.05}(-F) = -\int_{-\infty}^{\Phi^{-1}(0.05)} \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} dx = 2.06$$

(Φ is the distribution function of the standard normal law), which is unacceptably high for a buyer of such a contract (indeed, he/she will suffer a loss in 98% cases). On the other hand, an agent using the RAROC-based NGD pricing would charge the price

$$\sup_{\mathsf{Q}\in\frac{1}{1+R}\mathsf{P}+\frac{R}{1+R}\mathcal{D}} \mathsf{E}_{\mathsf{Q}}F = \frac{R}{1+R}\sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}F = \frac{0.1\cdot 2.06}{1.1} = 0.19,$$

which is reasonable. □

Thus, if we take as a risk measure one of those that are really used to measure *risk*, we should use the RAROC-based NGD. For the utility-based NGD, one should take from the outset much more "moderate" risk measure, like Tail V@R of order close to 1.

3.3 Static Model with a Finite Number of Assets

We consider the model of the previous subsection with $A = \{\langle h, S_1 - S_0\rangle : h \in \mathbb{R}^d\}$, where $S_0 \in \mathbb{R}^d$ and $S_1^1, \dots, S_1^d \in L_s^1(\mathcal{RD})$. From the financial point of view, S_n^i is the discounted price of the i-th asset at time n.

Clearly, in this model A is $\mathcal{RD}$-consistent and $\mathcal{RD} \cap \mathcal{R} = \mathcal{RD} \cap \mathcal{M}$, where $\mathcal{M}$ is the set of martingale measures:

$$\mathcal{M} = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E_Q}|S_1| < \infty \text{ and } \mathsf{E_Q} S_1 = S_0\}.$$

`Remark`. We have $\mathcal{M} \subseteq \mathcal{R}$, but the reverse inclusion might be violated. Indeed, let $d = 1$ and let S_1 be such that $\mathsf{E_P} S_1^+ = \mathsf{E_P} S_1^- = \infty$. Then $\mathsf{P} \in \mathcal{R}$, while $\mathsf{P} \notin \mathcal{M}$.

Let us now provide a geometric interpretation of Theorem 3.4 and Corollary 3.10. For this, we only assume that $\mathcal{PD} \subseteq \mathcal{RD} \subseteq \mathcal{P}$ are convex sets and $S_1 \in L_w^1(\mathcal{RD})$. Let us introduce the notation (see Figure 3)

$$\begin{aligned} E &= \mathrm{cl}\{\mathsf{E_Q} S_1 : \mathsf{Q} \in \mathcal{PD}\},\\ G &= \mathrm{cl}\{\mathsf{E_Q} S_1 : \mathsf{Q} \in \mathcal{RD}\},\\ G_R &= \frac{1}{1+R}\,E + \frac{R}{1+R}\,G,\\ D &= \mathrm{conv}\,\mathrm{supp}\,\mathrm{Law}_{\mathsf{P}}\, S_1, \end{aligned}$$

where "supp" denotes the support, and let D° denote the relative interior of D (i.e. the interior in the relative topology of the smallest affine subspace containing D). It is easy to see from the equalities

$$\begin{aligned} \inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E_Q}\langle h, S_1 - S_0\rangle &= \inf_{x \in E}\langle h, x - S_0\rangle,\\ \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E_Q}\langle h, S_1 - S_0\rangle &= \inf_{x \in G}\langle h, x - S_0\rangle \end{aligned}$$

that the following equivalences are true:

$$\begin{aligned} \text{RAROC-based NGD} &\iff S_0 \in G_R,\\ \text{utility-based NGD} &\iff S_0 \in G,\\ \text{NA} &\iff S_0 \in D^\circ, \end{aligned}$$

where the utility-based NGD employs $\mathcal{D} = \mathcal{RD}$ (the last equivalence is a well-known result of arbitrage pricing; see [64; Ch. V, § 2e]).

Now, let $F \in L_w^1(\mathcal{RD})$ be the discounted payoff of a contingent claim. Let $\widetilde{E}$, $\widetilde{G}$, $\widetilde{G}_R$, $\widetilde{D}$, and $\widetilde{D}^\circ$ denote the versions of the sets E, G, G_R, D, and D° defined for $\widetilde{S}_1 = (S_1^1, \dots, S_1^d, F)$ instead of S_1. Let $I_{NGD\,(R)}(F)$ denote the RAROC-based NGD price interval, $I_{NGD}(F)$ denote the utility-based NGD price interval, and $I_{NA}(F)$ denote the NA price interval. Then

$$\begin{aligned} I_{NGD\,(R)}(F) &= \{x : (S_0, x) \in \widetilde{G}_R\},\\ I_{NGD}(F) &= \{x : (S_0, x) \in \widetilde{G}\},\\ I_{NA}(F) &= \{x : (S_0, x) \in \widetilde{D}^\circ\}. \end{aligned}$$

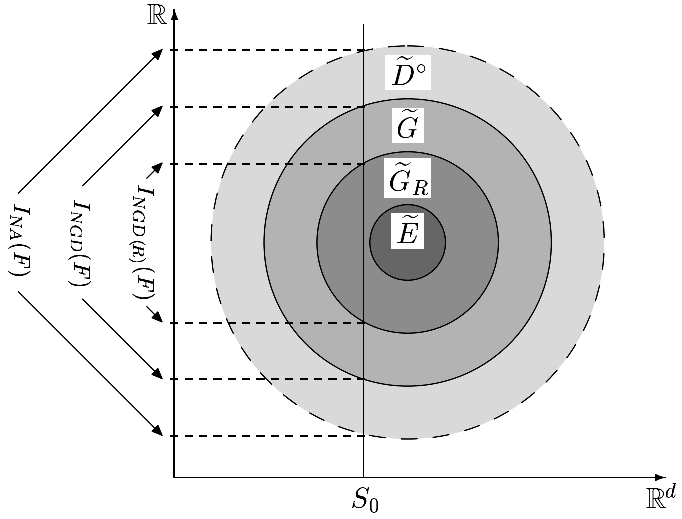

Figure 3. The geometric representation of price intervals provided by various techniques

Example 3.14. Let S_1 have Gaussian distribution with mean a and covariance matrix C. Let $\mathcal{PD} = \{\mathsf{P}\}$ and $\mathcal{RD}$ be the determining set of a law invariant coherent utility function u that is finite on Gaussian random variables. Let F be such that the vector $(S_1^1, \dots, S_1^d, F)$ is Gaussian. Denote $c = \mathsf{cov}(S_1, F)$ (we use the vector form of notation).

There exists $b \in \mathbb{R}^d$ such that $Cb = c$. We can write $F = \langle b, S_1 - a\rangle + \mathsf{E}F + \widetilde{F}$. Then $\mathsf{E}\widetilde{F} = 0$ and $\mathsf{cov}(\widetilde{F}, S_1) = 0$, so that $\widetilde{F}$ is independent of S_1. Note that

$$\sigma^2 := \mathsf{var}\,\widetilde{F} = \mathsf{var}\,F - \mathsf{var}\langle b, S_1 - a\rangle = \mathsf{var}\,F - \langle b, Cb\rangle = \mathsf{var}\,F - \langle b, c\rangle,$$

where var denotes the variance. Clearly, if $\sigma^2 = 0$, then

$$I_{NGD\,(R)}(F) = I_{NGD}(F) = I_{NA}(F) = \{\langle b, S_0 - a\rangle + \mathsf{E}F\}.$$

Let us now assume that $\sigma^2 > 0$.

Obviously, $I_{NA}(F) = \mathbb{R}$.

In order to find $I_{NGD}(F)$, note that $I_{NGD}(F) = \langle b, S_0 - a\rangle + \mathsf{E}F + I_{NGD}(\widetilde{F})$. Let L denote the image of $\mathbb{R}^d$ under the map $x \mapsto Cx$. Then the inverse $C^{-1} : L \to L$ is correctly defined. As u is law invariant, there exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. From this, it is easy to see that the set $\widetilde{G} := \{\mathsf{E}_\mathsf{Q}(S_1, \widetilde{F}) : \mathsf{Q} \in \mathcal{RD}\}$ has the form

$$\widetilde{G} = (a, 0) + \{(x, y) : x \in L,\ y \in \mathbb{R} : \langle x, C^{-1}x\rangle + \sigma^{-2}y^2 \le \gamma^2\}. \tag{3.3}$$

Consequently,

$$I_{NGD}(F) = \big[\langle b, S_0 - a\rangle + \mathsf{E}F - \alpha, \langle b, S_0 - a\rangle + \mathsf{E}F + \alpha\big],$$

where $\alpha = (\sigma^2\gamma^2 - \sigma^2\langle S_0 - a, C^{-1}(S_0 - a)\rangle)^{1/2}$. (In particular, the NGD is satisfied if and only if $\langle S_0 - a, C^{-1}(S_0 - a)\rangle \le \gamma^2$.)

Similar arguments show that

$$I_{NGD\,(R)}(F) = \big[\langle b, S_0 - a\rangle + \mathsf{E}F - \alpha(R), \langle b, S_0 - a\rangle + \mathsf{E}F + \alpha(R)\big],$$

where $\alpha(R) = \left(\frac{\sigma^2\gamma^2 R^2}{(1+R)^2} - \sigma^2\langle S_0 - a, C^{-1}(S_0 - a)\rangle\right)^{1/2}$. In particular, the NGD(R) condition is satisfied if and only if $\langle S_0 - a, C^{-1}(S_0 - a)\rangle \le \frac{\gamma^2 R^2}{(1+R)^2}$.

Let us remark that $I_{NGD}(F)$ and $I_{NGD(R)}(F)$ depend on u rather weakly, i.e. they depend only on γ. □

3.4 Dynamic Model with an Infinite Number of Assets

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is trivial. Let $\mathcal{D} \subseteq \mathcal{P}$ be a convex weakly compact set (in the framework of Subsection 3.1, $\mathcal{D}$ is the determining set of u; in the framework of Subsection 3.2, $\mathcal{D} = \frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}$). Let (S^i), $i \in I$ be a family of $(\mathcal{F}_t)$-adapted càdlàg processes (the set I is arbitrary, and we impose no assumptions on the probabilistic structure of S^i like the assumption that S^i is a semimartingale). From the financial point of view, S^i is the discounted price process of the i-th asset. We assume that $S^i_t \in L^1_s(\mathcal{D})$ for any $t \in [0,T]$, $i \in I$. The set of P&Ls that an agent can obtain by piecewise constant trading strategies (and only such strategies can be employed in practice) is naturally defined as

$$\begin{aligned} A = \Bigg\{ &\sum_{n=1}^{N}\sum_{i\in I} H^i_n(S^i_{u_n} - S^i_{u_{n-1}}) : N \in \mathbb{N},\ u_0 \le \dots \le u_N \text{ are } (\mathcal{F}_t)\text{-stopping times,} \\ &H^i_n \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H^i_n = 0 \text{ for all } i, \text{ except for a finite set}\Bigg\}. \end{aligned} \tag{3.4}$$

The following lemma shows that in this model A is $\mathcal{D}$-consistent and also provides the explicit form of the set $\mathcal{D} \cap \mathcal{R}$.

Lemma 3.15. *We have $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A') = \mathcal{D} \cap \mathcal{M}$, where*

$$\begin{aligned} A' &= \{H(S^i_v - S^i_u) : u \le v \in [0,T],\ i \in I,\ H \text{ is } \mathcal{F}_u\text{-measurable and bounded}\}, \\ \mathcal{M} &= \{\mathsf{Q} \in \mathcal{P} : \text{for any } i \in I,\ S^i \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\}. \end{aligned}$$

Proof. The inclusions $\mathcal{D} \cap \mathcal{R} \subseteq \mathcal{D} \cap \mathcal{R}(A') \subseteq \mathcal{D} \cap \mathcal{M}$ are clear. So, it is sufficient to prove the inclusion $\mathcal{D} \cap \mathcal{M} \subseteq \mathcal{D} \cap \mathcal{R}$. Let $\mathsf{Q} \in \mathcal{D} \cap \mathcal{M}$. Take $X = \sum_{n=1}^N \sum_{i\in I} H^i_n(S^i_{u_n} - S^i_{u_{n-1}}) \in A$. The process

$$M_k = \sum_{n=1}^{k}\sum_{i\in I} H^i_n(S^i_{u_n} - S^i_{u_{n-1}}), \quad k = 0, \dots, N$$

is an $(\mathcal{F}_{u_k}, \mathsf{Q})$-local martingale. Suppose that $\mathsf{E}_\mathsf{Q} X^- < \infty$ (otherwise, $\mathsf{E}_\mathsf{Q} X = -\infty$). Then M is a martingale (see [64; Ch. II, § 1c]), and hence, $\mathsf{E}_\mathsf{Q} X = \mathsf{E}_\mathsf{Q} M_N = 0$. Thus, in any case, $\mathsf{E}_\mathsf{Q} X \le 0$, which proves that $\mathsf{Q} \in \mathcal{R}$. □

Example 3.16. Let us consider the Black-Scholes-Merton model in the framework of the RAROC-based pricing. Thus, $S_t = S_0 e^{\mu t + \sigma B_t}$, $\mathcal{F}_t = \mathcal{F}^B_t$, where B is a Brownian motion; $\mathcal{PD} = \{\mathsf{P}\}$, and we are given a risk-determining set $\mathcal{RD}$. Surprisingly enough, in this model $\sup_{X\in A} \text{RAROC}(X) = \infty$. Indeed, the set $\mathcal{M}$ consists of a unique measure Q_0 and $\frac{d\mathsf{Q}_0}{d\mathsf{P}}$ is not bounded away from zero, so that condition (3.2) is violated for any $R > 0$.

Let us construct explicitly a sequence $X_n \in A$ with $\text{RAROC}(X_n) \to \infty$. Consider $D_n = \left\{\frac{d\mathsf{Q}_0}{d\mathsf{P}} < n^{-1}\right\}$ and set $X_n = a_n I(D_n) - I(\Omega \setminus D_n)$, where a_n is chosen in such a way that $\mathsf{E}_{\mathsf{Q}_0} X_n = 0$. Then $\mathsf{E}_\mathsf{P} X_n \to \infty$, while $\inf_{\mathsf{Q}\in\mathcal{RD}} \mathsf{E}_\mathsf{Q} X \ge -1$, so that $\mathrm{RAROC}(X_n) \to \infty$. Actually, $X_n \notin A$, but, for each n, there exists a sequence $(Y_n^m) \in A$ such that $-2 \le Y_n^m \le a_n + 1$ and $Y_n^m \xrightarrow[m\to\infty]{\mathsf{P}} X_n$. Then $\mathrm{RAROC}(Y_n^m) \xrightarrow[m\to\infty]{} \mathrm{RAROC}(X_n)$, so that $\mathrm{RAROC}\big(Y_n^{m(n)}\big) \to \infty$ for some subsequence $m(n)$.

This example shows that complete models are typically inconsistent with the RAROC-based NGD pricing. But this technique is primarily aimed at incomplete models because in complete ones the NA price intervals are already exact.

Let us also remark that the utility-based NGD condition might be naturally satisfied in the Black-Scholes-Merton model under an appropriate choice of the coherent utility u. □

3.5 Dynamic Model with Transaction Costs

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is trivial and $(\mathcal{F}_t)$ is right-continuous. Let $\mathcal{D} \subseteq \mathcal{P}$ be a convex weakly compact set. Let S^{ai}, S^{bi}, $i \in I$ be two families of $(\mathcal{F}_t)$-adapted càdlàg processes. From the financial point of view, S^{ai} (resp., S^{bi}) is the discounted ask (resp., bid) price process of the i-th asset, so that $S^a \ge S^b$ componentwise. We assume that $S_t^{ai}, S_t^{bi} \in L_s^1(\mathcal{D})$ for any $t \in [0,T]$, $i \in I$. The set of P&Ls that can be obtained in this model is naturally defined as

$$\begin{aligned} A = \bigg\{ &\sum_{n=0}^{N} \sum_{i\in I} \big[-H_n^i I(H_n^i > 0) S_{u_n}^{ai} - H_n^i I(H_n^i < 0) S_{u_n}^{bi}\big] : \\ &N \in \mathbb{N},\ u_0 \le \dots \le u_N \text{ are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_n}\text{-measurable,} \\ &H_n^i = 0 \text{ for all } i, \text{ except for a finite set, and } \sum_{n=0}^{N} H_n^i = 0 \text{ for any } i \bigg\}. \end{aligned}$$

Here H_n^i means the amount of the i-th asset that is bought at time u_n (so that $\sum_{k=0}^n H_k^i$ is the total amount of the i-th asset held at time u_n). Note that if there are no transaction costs, i.e. $S^{ai} = S^{bi} = S^i$ for each i, then A coincides with the set given by (3.4).

The following lemma shows that in this model A is $\mathcal{D}$-consistent and also provides the explicit form of the set $\mathcal{D} \cap \mathcal{R}$. Recall that a stopping time is simple if it takes on a finite number of values.

Lemma 3.17. *We have* $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A') = \mathcal{D} \cap \mathcal{M}$, *where*

$$\begin{aligned} A' = \{&G(S_v^{bi} - S_u^{ai}) + H(-S_v^{ai} + S_u^{bi}) : i \in I,\ u \le v \text{ are simple } (\mathcal{F}_t)\text{-} \\ &\text{stopping times, } G \text{ and } H \text{ are positive, bounded, } \mathcal{F}_u\text{-measurable}\}, \\ \mathcal{M} = \{&\mathsf{Q} \in \mathcal{P} : \text{for any } i, \text{ there exists an } (\mathcal{F}_t, \mathsf{Q})\text{-} \\ &\text{martingale } M^i \text{ such that } S^{bi} \le M^i \le S^{ai}\}. \end{aligned}$$

`Proof`. The inclusion $\mathcal{D} \cap \mathcal{R} \subseteq \mathcal{D} \cap \mathcal{R}(A')$ is obvious.

Let us prove the inclusion $\mathcal{D} \cap \mathcal{R}(A') \subseteq \mathcal{D} \cap \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}(A')$. Fix $i \in I$. For any simple stopping times $u \le v$, we have $S_u^{ai}, S_u^{bi}, S_v^{ai}, S_v^{bi} \in L_s^1(\mathcal{D})$ and

$$\mathsf{E}_{\mathsf{Q}}(S_v^{ai} \mid \mathcal{F}_u) \ge S_u^{bi}, \qquad \mathsf{E}_{\mathsf{Q}}(S_v^{bi} \mid \mathcal{F}_u) \le S_u^{ai}. \tag{3.5}$$

Consider the Snell envelopes

$$\begin{aligned} X_t &= \operatorname*{esssup}_{\tau\in\mathcal{T}_t} \mathsf{E}_{\mathsf{Q}}(S_\tau^{bi} \mid \mathcal{F}_t), \quad t \in [0,T], \\ Y_t &= \operatorname*{essinf}_{\tau\in\mathcal{T}_t} \mathsf{E}_{\mathsf{Q}}(S_\tau^{ai} \mid \mathcal{F}_t), \quad t \in [0,T], \end{aligned}$$

where $\mathcal{T}_t$ denotes the set of simple $(\mathcal{F}_t)$-stopping times such that $\tau \ge t$. (Recall that $\operatorname{esssup}_\alpha \xi_\alpha$ is a random variable ξ such that, for any α, $\xi \ge \xi_\alpha$ a.s. and for any other random variable ξ' with this property, we have $\xi \le \xi'$ a.s.) Then X is an $(\mathcal{F}_t)$-supermartingale, while Y is an $(\mathcal{F}_t, \mathsf{Q})$-submartingale (see [30; Th. 2.12.1]).

Let us prove that, for any $t \in [0,T]$, $X_t \le Y_t$ Q-a.s. Assume that there exists t such that $\mathsf{P}(X_t > Y_t) > 0$. Then there exist $\tau, \sigma \in \mathcal{T}_t$ such that

$$\mathsf{Q}\big(\mathsf{E}_\mathsf{Q}(S_\tau^{bi} \mid \mathcal{F}_t) > \mathsf{E}_\mathsf{Q}(S_\sigma^{ai} \mid \mathcal{F}_t)\big) > 0.$$

This implies that $\mathsf{Q}(\xi > \eta) > 0$, where $\xi = \mathsf{E}_\mathsf{Q}(S_\tau^{bi} \mid \mathcal{F}_{\tau\wedge\sigma})$ and $\eta = \mathsf{E}_\mathsf{Q}(S_\sigma^{ai} \mid \mathcal{F}_{\tau\wedge\sigma})$. Assume first that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \le \sigma\}) > 0$. On the set $\{\tau \le \sigma\}$ we have

$$\xi = S_\tau^{bi} = S_{\tau\wedge\sigma}^{bi}, \qquad \eta = \mathsf{E}_\mathsf{Q}(S_\sigma^{ai} \mid \mathcal{F}_{\tau\wedge\sigma}) = \mathsf{E}_\mathsf{Q}(S_{\tau\vee\sigma}^{ai} \mid \mathcal{F}_{\tau\wedge\sigma}),$$

and we obtain a contradiction with (3.5). In a similar way we get a contradiction if we assume that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \ge \sigma\}) > 0$. As a result, $X_t \le Y_t$ Q-a.s. Now, it follows from [42; Lem. 3] that there exists an $(\mathcal{F}_t, \mathsf{Q})$-martingale M such that $X \le M \le Y$. As a result, $\mathsf{Q} \in \mathcal{M}$.

Let us prove the inclusion $\mathcal{D} \cap \mathcal{M} \subseteq \mathcal{D} \cap \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{D} \cap \mathcal{M}$, so that, for any i, there exists an $(\mathcal{F}_t, \mathsf{Q})$-martingale M^i such that $S^{bi} \le M^i \le S^{ai}$. For any

$$X = \sum_{n=0}^N \sum_{i\in I} \big[-H_n^i I(H_n^i > 0) S_{u_n}^{ai} - H_n^i I(H_n^i < 0) S_{u_n}^{bi}\big] \in A,$$

we have

$$X \le \sum_{n=0}^N \sum_{i\in I} \big[-H_n^i I(H_n^i > 0) M_{u_n}^i - H_n^i I(H_n^i < 0) M_{u_n}^i\big] = \sum_{n=1}^N \sum_{i\in I} \Big(\sum_{k=0}^{n-1} H_k^i\Big)(M_{u_n}^i - M_{u_{n-1}}^i).$$

Repeating the arguments used in the proof of Lemma 3.15, we get $\mathsf{E}_\mathsf{Q} X \le 0$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

Consider now a model with proportional transaction costs, i.e. $S^{ai} = S^i$, $S^{bi} = (1-\lambda^i) S^i$, where each S^i is positive, while $\lambda^i \in (0,1)$. Denote the interval of the NGD prices in this model by $I_\lambda(F)$ (the NGD pricing technique might be utility-based or RAROC-based as the latter one is reduced to the former one by considering $\mathcal{D} = \frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}$). Let $(\lambda_n) = (\lambda_n; i \in I, n \in \mathbb{N})$ be a sequence such that $\lambda_n^i \xrightarrow[n\to\infty]{} 0$ for any i.

Theorem 3.18. *For $F \in L_s^1(\mathcal{D})$, we have $I_{\lambda_n}(F) \xrightarrow[n\to\infty]{} I_0(F)$ in the sense that the right (resp., left) endpoints of $I_{\lambda_n}(F)$ converge to the right (resp., left) endpoint of $I_0(F)$.*

Proof. Let r denote the right endpoint of $I_0(F)$. Suppose that the right endpoints of $I_{\lambda_n}(F)$ do not converge to r. As $I_0(F) \subseteq I_{\lambda_n}(F)$, there exists $r' > r$ such that, for each n (possibly, after passing on to a subsequence), there exists $\mathsf{Q}_n \in \mathcal{D} \cap \mathcal{R}_{\lambda_n}$ with the property: $\mathsf{E}_\mathsf{Q} F \ge r'$ ($\mathcal{R}_\lambda$ is the set of risk-neutral measures in the model corresponding to λ). The sequence (Q_n) has a weak limit point $\mathsf{Q}_\infty \in \mathcal{D}$. Fix $i \in I$, $u \le v \in [0,T]$, and a positive bounded $\mathcal{F}_u$-measurable random variable H. For any n, we have $\mathsf{E}_{\mathsf{Q}_n} H((1-\lambda_n^i) S_v^i - S_u^i) \le 0$. As $S_v^i \in L_s^1(\mathcal{D})$, we have $\sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} S_v^i < \infty$, and hence, $\limsup_n \mathsf{E}_{\mathsf{Q}_n} H(S_v^i - S_u^i) \le 0$. As the map $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} H(S_v^i - S_u^i)$ is weakly continuous, we get $\mathsf{E}_{\mathsf{Q}_\infty} H(S_v^i - S_u^i) \le 0$. In a similar way, we prove that $\mathsf{E}_{\mathsf{Q}_\infty} H(-S_v^i + S_u^i) \le 0$. Thus, S^i is an $(\mathcal{F}_t, \mathsf{Q}_\infty)$-martingale, so that $\mathsf{Q}_\infty \in \mathcal{D} \cap \mathcal{R}_0$. As the map $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} F$ is weakly continuous, we should have $\mathsf{E}_{\mathsf{Q}_\infty} F \ge r'$. But this is a contradiction. □

3.6 Hedging

Consider the model of Subsection 3.1.

Definition 3.19. The *upper and lower NGD prices* of a contingent claim F are defined by

$$\overline{V}(F) = \inf\{x : \exists X \in A \text{ such that } u(X - F + x) \geq 0\},$$
$$\underline{V}(F) = \sup\{x : \exists X \in A \text{ such that } u(X + F - x) \geq 0\}.$$

The problem of finding $\overline{V}(F)$ has some similarities with the superreplication problem considered by Cvitanić, Karatzas [18] and by Sekine [63], but the difference is that in those papers the risk is measured not as $\rho(X - F + x)$, but rather as $\rho((X - F + x)^-)$.

Proposition 3.20. *If A is a cone and $F \in L^1_s(\mathcal{D})$, then*

$$\overline{V}(F) = \sup\{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\},$$
$$\underline{V}(F) = \inf\{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}.$$

`Proof`. Take $x_0 \in \mathbb{R}$ and set $A(x_0) = A + \{h(x_0 - F) : h \in \mathbb{R}_+\}$. Using Theorem 3.4, we can write

$$\begin{aligned}\overline{V}(F) \geq x_0 &\iff \nexists X \in A \text{ such that } u(X - F + x_0) > 0\\ &\iff \nexists X \in A(x_0) \text{ such that } u(X) > 0\\ &\iff \mathcal{D} \cap \mathcal{R}(A(x_0)) \neq \emptyset\\ &\iff \exists \mathsf{Q} \in \mathcal{D} \cap \mathcal{R} \text{ such that } \mathsf{E_Q} F \geq x_0.\end{aligned}$$

This yields the formula for $\overline{V}(F)$. The representation of $\underline{V}(F)$ is proved similarly. □

`Remarks`. (i) The above statement is formally true if the NGD is violated. In this case $\overline{V}(F) = -\infty$ and $\underline{V}(F) = \infty$.

(ii) The above argument shows that there exist $\overline{\mathsf{Q}}, \underline{\mathsf{Q}} \in \mathcal{D} \cap \mathcal{R}$ such that $\mathsf{E}_{\overline{\mathsf{Q}}} F = \overline{V}(F)$, $\mathsf{E}_{\underline{\mathsf{Q}}}(F) = \underline{V}(F)$. This is in contrast with the NA technique. □

(iii) Under the conditions of the above proposition, we have $I_{NGD}(F) = [\underline{V}(F), \overline{V}(F)]$.

Let us now study the sub- and super-replication problem for a particular case of a (frictionless) static model with a finite number of assets. Thus, we are given a convex set $\mathcal{D} \subseteq \mathcal{P}$, $S_0 \in \mathbb{R}^d$, and $S^1_1, \dots, S^d_1 \in L^1_w(\mathcal{D})$. From the financial point of view, S^i_n is the discounted price of the i-th asset at time n.

Definition 3.21. The *superhedging and subhedging strategies* are defined by

$$\overline{H}(F) = \{h \in \mathbb{R}^d : u(\langle h, S_1 - S_0 \rangle - F + \overline{V}(F)) \geq 0\},$$
$$\underline{H}(F) = \{h \in \mathbb{R}^d : u(\langle h, S_1 - S_0 \rangle + F - \underline{V}(F)) \geq 0\}.$$

Below we provide a simple geometric procedure to determine these quantities. Assume that $F \in L^1_w(\mathcal{D})$ and let us introduce the notation

$$\begin{aligned}G &= \operatorname{cl}\{\mathsf{E_Q}(S_1, F) : \mathsf{Q} \in \mathcal{D}\},\\ \overline{v} &= \sup\{x : (S_0, x) \in G\},\\ \underline{v} &= \inf\{x : (S_0, x) \in G\},\\ \overline{N} &= \{h \in \mathbb{R}^{d+1} : \forall x \in G,\ \langle h, x - (S_0, \overline{v})\rangle \geq 0\},\\ \underline{N} &= \{h \in \mathbb{R}^{d+1} : \forall x \in G,\ \langle h, x - (S_0, \underline{v})\rangle \geq 0\},\end{aligned}$$

i.e. G is the generator for (S_1, F) and u; $\overline{N}$ (resp., $\underline{N}$) is the set of inner normals to G at the point $(S_0, \overline{v})$ (resp., $(S_0, \underline{v})$); see Figure 4.

Proposition 3.22. *We have*

$$\begin{aligned}\overline{V}(F) &= \overline{v},\\ \underline{V}(F) &= \underline{v},\\ \overline{H} &= \{h \in \overline{N} : h^{d+1} = -1\},\\ \underline{H} &= \{h \in \underline{N} : h^{d+1} = 1\}.\end{aligned}$$

`Remark`. The statement is formally true if the NGD is violated. In this case S_0 does not belong to the projection of G on $\mathbb{R}^d$, $\overline{v} = \overline{V}(F) = -\infty$, $\underline{v} = \underline{V}(F) = \infty$, $\overline{N} = \overline{H} = \emptyset$, $\underline{N} = \underline{H} = \emptyset$.

`Proof of Proposition 3.22.` We will prove the statement for $\underline{V}(F)$ and $\underline{H}$. Clearly, $\underline{V}(F) = \sup_h u(\langle h, S_1 - S_0\rangle + F)$ and $\underline{H} = \operatorname{argmax}_h u(\langle h, S_1 - S_0\rangle + F)$. We have

$$u(\langle h, S_1 - S_0\rangle + F) = \inf_{z\in G}\langle (h,1), z - (S_0, 0)\rangle \le \langle (h,1), (S_0, \underline{v}) - (S_0, 0)\rangle = \underline{v}, \quad h \in \mathbb{R}^d.$$

On the other hand, for $h \in \underline{H}$ (and only for such h), the above inequality becomes an equality. □

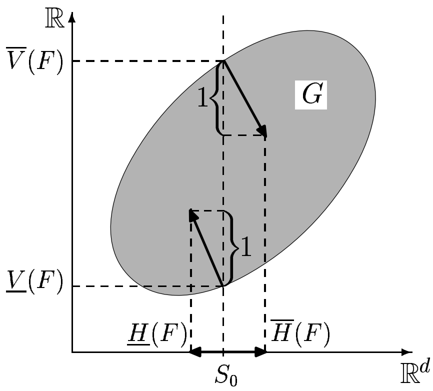

Figure 4. Solution of the super- and subhedging problem

Example 3.23. Consider the setting of Example 3.14. The results of that example show that

$$\begin{aligned}\overline{V}(F) &= \langle b, S_0 - a\rangle + \mathsf{E}F + \alpha,\\ \underline{V}(F) &= \langle b, S_0 - a\rangle + \mathsf{E}F - \alpha,\end{aligned}$$

where $\alpha = (\sigma^2\gamma^2 - \sigma^2\langle S_0 - a, C^{-1}(S_0 - a)\rangle)^{1/2}$. In order to find $\overline{H}$ and $\underline{H}$, we express the upper and lower borders of the set $\widetilde{G}$ given by (3.3) as $y = \pm(\sigma^2\gamma^2 - \sigma^2\langle x - a, C^{-1}(x - a)\rangle)^{1/2}$. Then by differentiation we get

$$\overline{H}(\widetilde{F}) = \underline{H}(\widetilde{F}) = d|_{x=S_0}(\sigma^2\gamma^2 - \sigma^2\langle x - a, C^{-1}(x - a)\rangle)^{1/2} = -\sigma^2\alpha^{-1}C^{-1}(S_0 - a).$$

Hence,

$$\begin{aligned}\overline{H}(F) &= b - \sigma^2\alpha^{-1}C^{-1}(S_0 - a),\\ \underline{H}(F) &= -b - \sigma^2\alpha^{-1}C^{-1}(S_0 - a).\end{aligned}$$

We will now provide one more important example. Let $S_0 \in (0,\infty)$ and S_1 be an integrable random variable such that $\operatorname{Law} S_1$ has no atoms and $\operatorname{supp}\operatorname{Law} S_1 = \mathbb{R}_+$. Let u be the coherent utility function corresponding to Tail V@R of order $\lambda \in (0,1]$ (see Example 2.5 (i)). We assume that $u(S_1) < S_0 < -u(-S_1)$. Finally, let $F = f(S_1)$, where $f : \mathbb{R}_+ \to \mathbb{R}$ is a convex function of linear growth. Let us denote $\operatorname{Law} S_1$ by Q and find a, b, c, d such that $a+b=\lambda$, $d-c=\lambda$, and

$$\begin{aligned}&\lambda^{-1}\int_0^{q_a} x\mathsf{Q}(dx) + \lambda^{-1}\int_{q_{1-b}}^{\infty} x\mathsf{Q}(dx) = S_0,\\ &\lambda^{-1}\int_{q_c}^{q_d} x\mathsf{Q}(dx) = S_0,\end{aligned}$$

where q_x is the x-quantile of Q.

Proposition 3.24. *We have*

$$\begin{aligned}\overline{V}(F) &= \lambda^{-1}\int_0^{q_a} f(x)\mathsf{Q}(dx) + \lambda^{-1}\int_{q_{1-b}}^{\infty} f(x)\mathsf{Q}(dx),\\ \underline{V}(F) &= \lambda^{-1}\int_{q_c}^{q_d} f(x)\mathsf{Q}(dx),\\ \overline{H}(F) &= \frac{f(q_{1-b})-f(q_a)}{q_{1-b}-q_a},\\ \underline{H}(F) &= -\frac{f(q_d)-f(q_c)}{q_d-q_c}.\end{aligned}$$

Proof. Let us first prove the representation for $\overline{V}(F)$ under an additional assumption that f is strictly convex. By Proposition 3.22,

$$\overline{V}(F) = \sup_{Z\in\mathcal{D}_\lambda:\, \mathsf{E}ZX=S_0} \mathsf{E}Zf(X)$$

($\mathcal{D}_\lambda$ is given by (2.3)). Take

$$Z_0 \in \operatorname*{argmax}_{Z\in\mathcal{D}_\lambda:\, \mathsf{E}ZX=S_0} \mathsf{E}Zf(X)$$

(Z_0 exists by a compactness argument). Passing from Z_0 to $\mathsf{E}(Z_0 \mid X)$, we can assume that Z_0 is X-measurable, i.e. $Z_0 = \varphi(X)$. Let us prove that

$$Z_0 = \lambda^{-1}I(X<q_a) + \lambda^{-1}I(X>q_{1-b}). \tag{3.6}$$

Assume the contrary. Then there exist $0<\alpha_1<\alpha_2<\alpha_3<\alpha_4$ such that

$$\begin{aligned}&\mathsf{Q}(\{\varphi<\lambda^{-1}\}\cap(\alpha_1,\alpha_2))>0,\\ &\mathsf{Q}(\{\varphi>0\}\cap(\alpha_2,\alpha_3))>0,\\ &\mathsf{Q}(\{\varphi<\lambda^{-1}\}\cap(\alpha_3,\alpha_4))>0.\end{aligned}$$

For $h_1,h_2,h_3 \in [0,\lambda^{-1}]$, we set

$$\widetilde{\varphi}(x) = \begin{cases}\varphi(x), & x\notin(\alpha_1,\alpha_4],\\ \varphi(x)\vee h_1, & x\in(\alpha_1,\alpha_2],\\ \varphi(x)\wedge h_2, & x\in(\alpha_2,\alpha_3],\\ \varphi(x)\vee h_3, & x\in(\alpha_3,\alpha_4].\end{cases}$$

We can find h_1, h_2, h_3 such that

$$\begin{gathered}
\mathsf{Q}(\{\widetilde{\varphi} > \varphi\} \cap (\alpha_1, \alpha_2)) > 0, \\
\mathsf{Q}(\{\widetilde{\varphi} < \varphi\} \cap (\alpha_2, \alpha_3)) > 0, \\
\mathsf{Q}(\{\widetilde{\varphi} > \varphi\} \cap (\alpha_3, \alpha_4)) > 0, \\
\int_0^\infty x\widetilde{\varphi}(x)\mathsf{Q}(dx) = \int_0^\infty x\varphi(x)\mathsf{Q}(dx) = S_0, \\
\int_0^\infty \widetilde{\varphi}(x)\mathsf{Q}(dx) = \int_0^\infty \varphi(x)\mathsf{Q}(dx) = 1.
\end{gathered}$$

Consider the affine function $\widetilde{f}$ that coincides with f at α_2 and α_3. Then

$$\int_0^\infty (\widetilde{\varphi}(x) - \varphi(x))\widetilde{f}(x)\mathsf{Q}(dx) = 0.$$

Furthermore, as f is strictly convex, $\widetilde{f} < f$ on (α_1, α_2), $\widetilde{f} > f$ on (α_2, α_3), and $\widetilde{f} > f$ on (α_3, α_4). Consequently,

$$\int_0^\infty (\widetilde{\varphi}(x) - \varphi(x))f(x)\mathsf{Q}(dx) > 0.$$

Thus, we have found $\widetilde{Z}_0 = \widetilde{\varphi}(X) \in \mathcal{D}_\lambda$ such that $\mathsf{E}\widetilde{Z}_0 X = S_0$ and $\mathsf{E}\widetilde{Z}_0 f(X) > \mathsf{E}Z_0 f(X)$, which contradicts the choice of Z_0. As a result, (3.6) is satisfied, which yields the desired representation of $\overline{V}(F)$.

Let us now prove the representation for $\overline{V}(F)$ in the general case. Take Z_0 given by (3.6). Find a strictly convex function $\widetilde{f}$ of linear growth. Then the function $f_\varepsilon = f + \varepsilon\widetilde{f}$ is strictly convex and the result proved above shows that $\mathsf{E}Zf_\varepsilon(X) \le \mathsf{E}Z_0 f_\varepsilon(X)$ for any $Z \in \mathcal{D}_\lambda$ such that $\mathsf{E}ZX = S_0$. Passing on to the limit as $\varepsilon \downarrow 0$, we get $\mathsf{E}Zf(X) \le \mathsf{E}Z_0 f(X)$ for any $Z \in \mathcal{D}_\lambda$ such that $\mathsf{E}ZX = S_0$. This yields the desired representation of $\overline{V}(F)$.

Let us now prove the representation for $\overline{H}(F)$. Consider the function

$$g(x) = \sup_{Z \in \mathcal{D}_\lambda :\, \mathsf{E}ZX = x} \mathsf{E}Zf(X), \quad x \in [u(S_1), -u(-S_1)].$$

It follows from the reasoning given above that $g = g_1 \circ g_2^{-1}$, where

$$\begin{aligned}
g_1(x) &= \lambda^{-1}\int_0^{q_x} f(y)\mathsf{Q}(dy) + \lambda^{-1}\int_{q_{1-\lambda+x}}^\infty f(y)\mathsf{Q}(dy), \quad x \in [0, \lambda^{-1}], \\
g_2(x) &= \lambda^{-1}\int_0^{q_x} y\mathsf{Q}(dy) + \lambda^{-1}\int_{q_{1-\lambda+x}}^\infty y\mathsf{Q}(dy), \quad x \in [0, \lambda^{-1}].
\end{aligned}$$

Applying Proposition 3.22, we get

$$\overline{H}(F) = g'(S_0) = \frac{f(q_{1-b}) - f(q_a)}{q_{1-b} - q_a}.$$

The representations for $\underline{V}(F)$ and $\underline{H}(F)$ are proved in a similar way. □

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