To my Teacher A.N. Shiryaev
on the occasion of his 70th birthday

SOME PARTICULAR PROBLEMS OF MARTINGALE THEORY

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Abstract. This paper deals with the following problems:

Is a product of independent martingales also a martingale? We consider 8 particular formulations of this problem.

Is a limit of a converging sequence of martingales also a martingale? We consider 32 particular formulations of this problem.

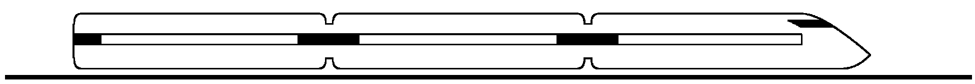
Is a stochastic integral of a bounded process with respect to a martingale also a martingale?

If $X = (X_t)_{t\ge 0}$ is a positive process such that $\mathsf{E}X_\tau = \mathsf{E}X_0$ for any finite stopping time τ, then is is true that X is a uniformly integrable martingale?

Key words and phrases. Convergence of martingales, local martingales, martingales, orthogonal local martingales, quadratic covariation, stochastic integrals, uniformly integrable martingales.

1 Introduction

The Seminar "Stochastic Analysis and Financial Mathematics" conducted at the Department of Probability Theory, Faculty of Mechanics and Mathematics, Moscow State University, by A.N. Shiryaev, A.A. Gushchin, M.A. Urusov, and the author is in some sense a continuation of the Seminar held at the Steklov Mathematical Institute in the 1970s and 1980s. The latter one was founded by A.N. Shiryaev in 1966 and was conducted by A.N. Shiryaev, N.V. Krylov, R.S. Liptser, and Yu.M. Kabanov. The new Seminar is sometimes called the "railroad seminar" because it is intended to work "as regularly as the railroad". The Seminar has its own symbol:



One of the distinctive features of this Seminar is that a particular problem is proposed to the listeners at each meeting and its solution is discussed at the next meeting. These are called "corner problems" because they are written at a corner of the blackboard.

In this paper, several such problems are considered. Some of the particular formulations are known or very easy to solve; some others are more complicated, and the obtained (negative or positive) results seem to be new.

1. Products of independent martingales. The problem is as follows: Is a product of independent martingales also a martingale? We consider 8 formulations of this problem:

1. Let X and Y be martingales (each with respect to its natural filtration). Is it true that XY is a martingale (with respect to its natural filtration)?

2. Let X and Y be local martingales (each with respect to its natural filtration). Is it true that XY is a local martingale (with respect to its natural filtration)?

3. Let X and Y be martingales with respect to a common filtration $(\mathcal{F}_t)$. Is it true that XY is an $(\mathcal{F}_t)$-martingale?

4. Let X and Y be local martingales with respect to a common filtration $(\mathcal{F}_t)$. Is it true that XY is an $(\mathcal{F}_t)$-local martingale?

5. Let X and Y be continuous martingales (each with respect to its natural filtration). Is it true that XY is a martingale (with respect to its natural filtration)?

6. Let X and Y be continuous local martingales (each with respect to its natural filtration). Is it true that XY is a local martingale (with respect to its natural filtration)?

7. Let X and Y be continuous martingales with respect to a common filtration $(\mathcal{F}_t)$. Is it true that XY is an $(\mathcal{F}_t)$-martingale?

8. Let X and Y be continuous local martingales with respect to a common filtration $(\mathcal{F}_t)$. Is it true that XY is an $(\mathcal{F}_t)$-local martingale?

Here the time index t for X and Y runs through the positive half-line or through a compact interval (clearly, the answers to the above problems are the same in these two cases).

`Remarks`. `(i)` By a local martingale we mean a process X, for which there exists a localizing sequence (τ_n) such that, for any n, the stopped process $(X_{t\wedge\tau_n})$ is a martingale. An alternative definition is that the process $(X_{t\wedge\tau_n} I(\tau_n > 0))$ should be a martingale. It is easy to check that the answers to the problems under consideration are the same for these two definitions.

`(ii)` Two $(\mathcal{F}_t)$-local martingales whose product is also an $(\mathcal{F}_t)$-local martingale are said to be orthogonal. Thus, Problem 4 (resp., Problem 8) can be reformulated as follows: does the independence of local martingales (resp., continuous local martingales) imply their orthogonality?

2. Limits of martingales. The problem is as follows: Is a limit of a converging sequence of martingales also a martingale? We consider 8 formulations of this problem:

1. Let (X^n) be a sequence of martingales (each with respect to its natural filtration) that converges to a process X in the sense of the weak convergence of finite-dimensional distributions. Is it true that X is a martingale (with respect to its natural filtration)?

2. Let (X^n) be a sequence of martingales (each with respect to its natural filtration) that converges in distribution to a process X. Is it true that X is a martingale (with respect to its natural filtration)?

3. Let (X^n) be a sequence of local martingales (each with respect to its natural filtration) that converges to a process X in the sense of the weak convergence of finite-dimensional distributions. Is it true that X is a local martingale (with respect to its natural filtration)?

4. Let (X^n) be a sequence of local martingales (each with respect to its natural filtration) that converges in distribution to a process X. Is it true that X is a local martingale (with respect to its natural filtration)?

5. Let (X^n) be a sequence of martingales with respect to a common filtration $(\mathcal{F}_t)$ such that $X_t^n \xrightarrow[n\to\infty]{\mathsf{P}} X_t$ for any t. Is it true that X is an $(\mathcal{F}_t)$-martingale?

6. Let (X^n) be a sequence of martingales with respect to a common filtration $(\mathcal{F}_t)$ that converges to a process X in probability uniformly on compact intervals (i.e. $\sup_{s\le t}|X_s^n - X_s| \xrightarrow[n\to\infty]{\mathsf{P}} 0$ for any t). Is it true that X is an $(\mathcal{F}_t)$-martingale?

7. Let (X^n) be a sequence of local martingales with respect to a common filtration $(\mathcal{F}_t)$ such that $X_t^n \xrightarrow[n\to\infty]{\mathsf{P}} X_t$ for any t. Is it true that X is an $(\mathcal{F}_t)$-local martingale?

8. Let (X^n) be a sequence of local martingales with respect to a common filtration $(\mathcal{F}_t)$ that converges to a process X in probability uniformly on compact intervals. Is it true that X is an $(\mathcal{F}_t)$-local martingale?

Here the time index t for X^n runs through the positive half-line or through a compact interval (clearly, the answers to the above problems are the same in these two cases).

We consider each of the above problems in combination with one of the following conditions on (X^n):

A. No additional assumptions on (X^n) are imposed.

B. The jumps of X^n are assumed to be bounded by a constant $a > 0$ and $X_0^n = 0$.

C. The processes X^n are assumed to be continuous and $X_0^n = 0$.

D. The processes X^n are assumed to be bounded by a constant $a > 0$.

Thus, we get $32 = 8 \times 4$ formulations of the above problem. In formulations 2.A, 2.B, 2.D, 4.A, 4.B, and 4.D, we consider the weak convergence in the space D of càdlàg functions, while in formulations 2.C and 4.C, we consider the weak convergence in the space C of continuous functions.

`Remark`. The above problem arises in connection with limit theorems for stochastic processes (see [2]).

3. Stochastic integrals with respect to a martingale. The problem is as follows: Let X be an $(\mathcal{F}_t)$-martingale and H be an $(\mathcal{F}_t)$-predictable process such that $|H| \le 1$. Is it true that the stochastic integral of H with respect to X is also an $(\mathcal{F}_t)$-martingale?

`Remark`. If the word “martingale” in the above problem is replaced by the word “semimartingale”, “$\mathcal{H}^p$-semimartingale” (see [6]), “sigma-martingale” (see [2; Ch. III, § 6e]), “local martingale”, or “$\mathcal{H}^p$-martingale” (see [3; Ch. I, § 5]), then, clearly, the answer is positive.

4. Uniform integrability of martingales. The problem is as follows: Let $X = (X_t)_{t\ge 0}$ be an $(\mathcal{F}_t)$-adapted càdlàg positive process such that $\mathsf{E}X_\tau = \mathsf{E}X_0 < \infty$ for any $(\mathcal{F}_t)$-stopping time τ that is finite a.s. Is it true that X is a uniformly integrable $(\mathcal{F}_t)$-martingale?

Remark. The origin of this problem lies in financial mathematics. Namely, let X be the discounted price process of some asset. Define the set of discounted incomes that can be obtained by trading this asset as:

$$\biggl\{\sum_{n=1}^{N} H_n(X_{v_n} - X_{u_n}) : N \in \mathbb{N},\ u_0 \le \dots \le u_N < \infty$$
$$\text{are } (\mathcal{F}_t)\text{-stopping times, } H_n \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable}\biggr\}.$$

As in [1], define the set of equivalent risk-neutral measures as the set of probability measures $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}}\xi^- \ge \mathsf{E}_{\mathsf{Q}}\xi^+$ for any $\xi \in A$ (here $\xi^- = (-\xi)\vee 0$, $\xi^+ = \xi \vee 0$; the expectations $\mathsf{E}_{\mathsf{Q}}\xi^-$ and $\mathsf{E}_{\mathsf{Q}}\xi^+$ are allowed to take on the value $+\infty$). It is easy to show that a measure $\mathsf{Q} \sim \mathsf{P}$ is a risk-neutral measure if and only if $\mathsf{E}_{\mathsf{Q}}X_\tau = \mathsf{E}_{\mathsf{Q}}X_0$ for any finite $(\mathcal{F}_t)$-stopping time τ. Thus, the above problem can be reformulated as follows: does the class of equivalent risk-neutral measures in the above model coincide with the class of equivalent uniformly integrable martingale measures?

The reader is invited to solve as many of the above 42 problems as possible.

2 Products of Independent Martingales

The answer to the problem "Is a product of independent martingales also a martingale?" in formulations 1 and 5 is positive as shown by the following theorem.

Theorem 2.1. *Let X and Y be independent martingales (each with respect to its natural filtration). Then XY is a martingale (with respect to its natural filtration).*

Proof. Fix $s \le t$. For any $A \in \mathcal{F}_s^X$ ($(\mathcal{F}_t^X)$ denotes the natural filtration of X, i.e. $\mathcal{F}_t^X = \sigma(X_s;\ s \le t)$) and $B \in \mathcal{F}_s^Y$, we have

$$\mathsf{E}(X_tY_tI_AI_B) = \mathsf{E}(X_tI_A)\mathsf{E}(Y_tI_B) = \mathsf{E}(X_sI_A)\mathsf{E}(X_sI_B) = \mathsf{E}(X_sY_sI_AI_B).$$

By the monotone class lemma,

$$\bigl\{C \in \mathcal{F}_s^X \vee \mathcal{F}_s^Y : \mathsf{E}(X_tY_tI_C) = \mathsf{E}(X_sY_sI_C)\bigr\} = \mathcal{F}_s^X \vee \mathcal{F}_s^Y.$$

Hence, $\mathsf{E}\bigl(X_tY_t \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y\bigr) = X_sY_s$, which implies that $\mathsf{E}\bigl(X_tY_t \mid \mathcal{F}_s^{XY}\bigr) = X_sY_s$. This is the desired statement. □

The example below shows that the answer to the problem in formulation 2 is negative. The example is given in the continuous time, but it is easy to provide also a discrete-time one.

Example 2.2. *Let B be a Brownian motion and ξ be a non-integrable random variable that is independent of B. Set*

$$H_t^0 = \frac{I(t<1)}{1-t}, \quad t \ge 0,$$
$$\tau = \inf\biggl\{t \ge 0 : \int_0^t H_s^0\,dB_s = \xi\biggr\},$$
$$H_t = H_t^0 I(t \le \tau), \quad t \ge 0,$$
$$X_t = \int_0^t H_s\,dB_s, \quad t \ge 0.$$

Let η be a random variable independent of X taking on values ± 1 with probability $1/2$. Set $Y_t = \eta I(t \ge 1)$, $t \ge 0$. Then X and Y are independent local martingales (each with respect to its natural filtration), but XY is not a local martingale (with respect to its natural filtration).

`Proof`. The first statement is clear. The second one follows from the property that for any $(\mathcal{F}_t^{XY})$-stopping time τ, we have $\{\tau < 1\} \in \bigvee_{t<1} \mathcal{F}_t^{XY} = \{\emptyset, \Omega\}$, while $X_1 = \xi$ is non-integrable. □

The next example shows that the answer to the problem in formulations 3 and 4 is negative.

Example 2.3. *Let ξ and η be independent random variables taking on the values ± 1 with probability $1/2$. Set*

$$X_t = \begin{cases} 0, & t < 1, \\ \xi, & t \ge 1, \end{cases} \qquad Y_t = \begin{cases} 0, & t < 1, \\ \eta, & t \ge 1, \end{cases} \qquad \mathcal{F}_t = \begin{cases} \sigma(\xi\eta), & t < 1, \\ \sigma(\xi, \eta), & t \ge 1. \end{cases}$$

Then X and Y are independent $(\mathcal{F}_t)$-martingales, but XY is not an $(\mathcal{F}_t)$-local martingale.

`Proof`. The first statement follows from the independence of ξ and $\xi\eta$ and the independence of η and $\xi\eta$. In order to prove the second one, notice that XY is not an $(\mathcal{F}_t)$-martingale. Being bounded, it is not an $(\mathcal{F}_t)$-local martingale. □

`Remark`. Examples 2.2 and 2.3 show that if we add the additional assumption that the jumps of X and Y are bounded, the answers to the problem in formulations 2, 3, and 4 will remain negative.

The theorem below shows that the answer to the problem in formulation 8 is positive.

Theorem 2.4. *Let X and Y be independent continuous $(\mathcal{F}_t)$-local martingales. Then XY is an $(\mathcal{F}_t)$-local martingale.*

`Proof`. Let us first assume that X and Y are bounded. Then, for any t and any sequence (Δ^n) of partitions of $[0, t]$ whose diameters tend to 0, we have

$$\begin{aligned} &\mathsf{E}\Big(\sum_{t_i \in \Delta^n} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})\Big)^2 \\ &= \sum_{t_i \in \Delta^n} \mathsf{E}(X_{t_{i+1}} - X_{t_i})^2 \mathsf{E}(Y_{t_{i+1}} - Y_{t_i})^2 \\ &\le \max_{t_i \in \Delta^n} \mathsf{E}(X_{t_{i+1}} - X_{t_i})^2 \cdot \sum_{t_i \in \Delta^n} \mathsf{E}(Y_{t_{i+1}} - Y_{t_i})^2 \\ &= \max_{t_i \in \Delta^n} (\mathsf{E}X_{t_{i+1}}^2 - \mathsf{E}X_{t_i}^2) \cdot (\mathsf{E}Y_t^2 - \mathsf{E}Y_0^2). \end{aligned}$$

The latter quantity tends to 0 as $n \to \infty$ since the function $s \mapsto \mathsf{E}X_s^2$ is continuous in s. Consequently, $\langle X, Y \rangle = 0$, which implies that XY is an $(\mathcal{F}_t)$-local martingale.

Consider now the general case. Set $\widetilde{X}_t = X_t - X_0$, $\widetilde{Y}_t = Y_t - Y_0$. Then

$$X_t Y_t = X_0 Y_0 + X_0 \widetilde{Y}_t + \widetilde{X}_t Y_0 + \widetilde{X}_t \widetilde{Y}_t.$$

For $n \in \mathbb{N}$, set $\tau_n = \inf\{t : |\widetilde{X}_t| \geq n\}$, $\sigma_n = \inf\{t : |\widetilde{Y}_t| \geq n\}$. Then the stopped processes $\widetilde{X}^{\tau_n} = (\widetilde{X}_{t\wedge\tau_n})$ and $\widetilde{Y}^{\sigma_n} = (\widetilde{Y}_{t\wedge\sigma_n})$ are independent $(\mathcal{F}_t)$-local martingales. Being bounded, they are $(\mathcal{F}_t)$-martingales. Clearly, $X_0\widetilde{Y}^{\sigma_n}$ and $\widetilde{X}^{\tau_n}Y_0$ are $(\mathcal{F}_t)$-martingales. By the reasoning above, $X^{\tau_n}Y^{\sigma_n}$ is an $(\mathcal{F}_t)$-local martingale. Being bounded, it is an $(\mathcal{F}_t)$-martingale. Consequently, for any $n \in \mathbb{N}$, $(XY)^{\tau_n\wedge\sigma_n}$ is an $(\mathcal{F}_t)$-martingale. As $\tau_n \wedge \sigma_n \xrightarrow[n\to\infty]{} \infty$, we get the desired statement. □

The next theorem shows that the answer to the problem in formulation 6 is positive.

Theorem 2.5. *Let X and Y be independent continuous local martingales (each with respect to its natural filtration). Then XY is a local martingale (with respect to its natural filtration).*

`Proof`. For $n \in \mathbb{N}$, set $\tau_n = \inf\{t : |X_t| \geq n\}$. Then the stopped process X^{τ_n} is an $(\mathcal{F}_t^X)$-local martingale. As $|X^{\tau_n}| \leq |X_0| \vee n$ and the latter random variable is integrable, the process X^{τ_n} is an $(\mathcal{F}_t^X)$-martingale. For any $s \leq t$, $A \in \mathcal{F}_s^X$, and $B \in \mathcal{F}_s^Y$, we have

$$\mathsf{E}(X_t^{\tau_n} I_A I_B) = \mathsf{E}(X_t^{\tau_n} I_A)\mathsf{P}(B) = \mathsf{E}(X_s^{\tau_n} I_A)\mathsf{P}(B) = \mathsf{E}(X_s^{\tau_n} I_A I_B).$$

Applying the monotone class lemma, we deduce that X^{τ_n} is a martingale with respect to the filtration $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^Y$. As τ_n is an $(\mathcal{F}_t)$-stopping time, X is an $(\mathcal{F}_t)$-local martingale. Similarly, Y is in the same class. By Theorem 2.4, XY is an $(\mathcal{F}_t)$-local martingale.

For $n \in \mathbb{N}$, set $\rho_n = \inf\{t : |X_t Y_t| \geq n\}$. Then $(XY)^{\rho_n}$ is an $(\mathcal{F}_t)$-local martingale. As $|(XY)^{\rho_n}| \leq |X_0 Y_0| \vee n$, the process $(XY)^{\rho_n}$ is an $(\mathcal{F}_t)$-martingale. Note that ρ_n is an $(\mathcal{F}_t^{XY})$-stopping time. Hence, XY is an $(\mathcal{F}_t^{XY})$-local martingale. □

The next theorem shows that the answer to the problem in formulation 7 is positive.

Theorem 2.6. *Let X and Y be independent continuous $(\mathcal{F}_t)$-martingales. Then XY is an $(\mathcal{F}_t)$-martingale.*

`Proof`. Set $\widetilde{X}_t = X_t - X_0$, $\widetilde{Y}_t = Y_t - Y_0$. Then

$$X_t Y_t = X_0 Y_0 + X_0 \widetilde{Y}_t + \widetilde{X}_t Y_0 + \widetilde{X}_t \widetilde{Y}_t,$$

and it is sufficient to prove that $\widetilde{X}\widetilde{Y}$ is an $(\mathcal{F}_t)$-martingale. Fix $s \leq t$. For $n \in \mathbb{N}$, set $\tau_n = \inf\{t : |\widetilde{X}_t| = n\}$ and $\sigma_n = \inf\{t : |\widetilde{Y}_t| = n\}$. Then the stopped processes $\widetilde{X}^{\tau_n}$ and $\widetilde{Y}^{\sigma_n}$ are independent continuous $(\mathcal{F}_t)$-martingales and, by Theorem 2.4, $\widetilde{X}^{\tau_n}\widetilde{Y}^{\sigma_n}$ is an $(\mathcal{F}_t)$-local martingale. Being bounded, it is an $(\mathcal{F}_t)$-martingale. Hence,

$$\mathsf{E}\big(\widetilde{X}_t^{\tau_n}\widetilde{Y}_t^{\sigma_n} \,\big|\, \mathcal{F}_s\big) = \widetilde{X}_s^{\tau_n}\widetilde{Y}_s^{\sigma_n}. \tag{2.1}$$

Furthermore, $\widetilde{X}_t^{\tau_n} \xrightarrow[n\to\infty]{\text{a.s.}} \widetilde{X}_t$ and the family $\big(\widetilde{X}_t^{\tau_n}\big)_{n\in\mathbb{N}}$ is uniformly integrable due to the martingale property of $\widetilde{X}$. Consequently, $\widetilde{X}_t^{\tau_n} \xrightarrow[n\to\infty]{L^1} \widetilde{X}_t$. Similarly, $\widetilde{Y}_t^{\sigma_n} \xrightarrow[n\to\infty]{L^1} \widetilde{Y}_t$. By the independence of $\widetilde{X}$ and $\widetilde{Y}$,

$$\begin{aligned}\mathsf{E}\big|\widetilde{X}_t^{\tau_n}\widetilde{Y}_t^{\sigma_n} - \widetilde{X}_t\widetilde{Y}_t\big| &\leq \mathsf{E}\big|\widetilde{X}_t^{\tau_n}\big(\widetilde{Y}_t^{\sigma_n} - \widetilde{Y}_t\big)\big| + \mathsf{E}\big|\big(\widetilde{X}_t^{\tau_n} - \widetilde{X}_t\big)\widetilde{Y}_t\big| \\ &= \mathsf{E}\big|\widetilde{X}_t^{\tau_n}\big| \cdot \mathsf{E}\big|\widetilde{Y}_t^{\sigma_n} - \widetilde{Y}_t\big| + \mathsf{E}\big|\widetilde{X}_t^{\tau_n} - \widetilde{X}_t\big| \cdot \mathsf{E}\big|\widetilde{Y}_t\big| \\ &\leq \mathsf{E}\big|\widetilde{X}_t\big| \cdot \mathsf{E}\big|\widetilde{Y}_t^{\tau_n} - \widetilde{Y}_t\big| + \mathsf{E}\big|\widetilde{X}_t^{\tau_n} - \widetilde{X}_t\big| \cdot \mathsf{E}\big|\widetilde{Y}_t\big| \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

(The last inequality is the Jensen inequality applied to the martingale $\widetilde{X}$.) Thus, $\widetilde{X}_t^{\tau_n}\widetilde{Y}_t^{\sigma_n} \xrightarrow[n\to\infty]{L^1} \widetilde{X}_t\widetilde{Y}_t$. Now, (2.1) implies that $\mathsf{E}(\widetilde{X}_t\widetilde{Y}_t \mid \mathcal{F}_s) = \widetilde{X}_s\widetilde{Y}_s$, which is the desired property. □

Formulation	Answer
1. $X \in \mathcal{M}, Y \in \mathcal{M}, X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}$	Yes, Theorem 2.1
2. $X \in \mathcal{M}_{\text{loc}}, Y \in \mathcal{M}_{\text{loc}}, X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}_{\text{loc}}$	No, Example 2.2
3. $X \in \mathcal{M}(\mathcal{F}_t), Y \in \mathcal{M}(\mathcal{F}_t), X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}(\mathcal{F}_t)$	No, Example 2.3
4. $X \in \mathcal{M}_{\text{loc}}(\mathcal{F}_t), Y \in \mathcal{M}_{\text{loc}}(\mathcal{F}_t), X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}_{\text{loc}}(\mathcal{F}_t)$	No, Example 2.3
5. $X \in \mathcal{M}^c, Y \in \mathcal{M}^c, X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}^c$	Yes, Theorem 2.1
6. $X \in \mathcal{M}^c_{\text{loc}}, Y \in \mathcal{M}^c_{\text{loc}}, X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}^c_{\text{loc}}$	Yes, Theorem 2.5
7. $X \in \mathcal{M}^c(\mathcal{F}_t), Y \in \mathcal{M}^c(\mathcal{F}_t), X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}^c(\mathcal{F}_t)$	Yes, Theorem 2.6
8. $X \in \mathcal{M}^c_{\text{loc}}(\mathcal{F}_t), Y \in \mathcal{M}^c_{\text{loc}}(\mathcal{F}_t), X \perp\!\!\!\perp Y \overset{?}{\Longrightarrow} XY \in \mathcal{M}^c_{\text{loc}}(\mathcal{F}_t)$	Yes, Theorem 2.4

`Table 1.` Summary of the answers to the problem "Is a product of independent martingales also a martingale?". Here we use the following notation: "$X \perp\!\!\!\perp Y$" means that X and Y are independent; "$X \in \mathcal{M}$" means that X is a martingale with respect to its natural filtration; "$X \in \mathcal{M}_{\text{loc}}$" means that X is a local martingale with respect to its natural filtration; "$X \in \mathcal{M}^c$" means that X is a continuous martingale with respect to its natural filtration; "$X \in \mathcal{M}(\mathcal{F}_t)$" means that X is an $(\mathcal{F}_t)$-martingale, and so on.

3 Limits of Martingales

The answer to the problem "Is a limit of a converging sequence of martingales also a martingale?" in formulations 1.A–8.A is negative as shown by the following example.

Example 3.1. *Let ξ be a non-integrable symmetric (i.e. $\xi \overset{\text{Law}}{=} -\xi$) random variable. Set*

$$X_t^n = \begin{cases} 0, & t < 1, \\ -n \vee \xi \wedge n, & t \geq 1, \end{cases} \qquad X_t = \begin{cases} 0, & t < 1, \\ \xi, & t \geq 1, \end{cases} \qquad \mathcal{F}_t = \mathcal{F}_t^X.$$

Then each X^n is a martingale with respect to its natural filtration as well as with respect to the filtration $(\mathcal{F}_t)$. Furthermore, (X^n) converges to X in probability uniformly on compact intervals (hence, the convergence in distribution also holds). However, X is not an $(\mathcal{F}_t)$-local martingale.

`Proof.` The first two statements are obvious. The last one follows from the property that for any $(\mathcal{F}_t^X)$-stopping time τ, we have $\{\tau < 1\} \in \bigvee_{t<1} \mathcal{F}_t^X = \{\emptyset, \Omega\}$. □

The next example shows that the answer to the problem in formulations 1.B, 1.C, 2.B, 2.C, 5.B, 5.C, 6.B, and 6.C is negative.

Example 3.2. *Let B be a 3-dimensional Brownian motion started at a point $B_0 \neq 0$. Set*

$$X_t = \frac{1}{\sqrt{(B_t^1)^2 + (B_t^2)^2 + (B_t^3)^2}}, \quad t \geq 0,$$
$$\tau_n = \inf\{t \geq 0 : X_t \geq n\},$$
$$X_t^n = X_{t\wedge\tau_n}, \quad t \geq 0.$$

Then each X^n is a continuous martingale with respect to its natural filtration as well as with respect to the filtration $\mathcal{F}_t = \mathcal{F}_t^X$. Furthermore, (X^n) converges to X in probability uniformly on compact intervals (hence, the convergence in distribution also holds). However, X is not a martingale with respect to any filtration.

`Proof`. By Itô's formula,

$$X_t = X_0 - \sum_{i=1}^{3} \int_0^t \frac{B_s^i}{((B_s^1)^2 + (B_s^2)^2 + (B_s^3)^2)^{3/2}}\, dB_s^i.$$

Therefore, X and each X^n are $(\mathcal{F}_t^B)$-local martingales. Being bounded, each X^n is an $(\mathcal{F}_t^B)$-martingale and hence, it also an $(\mathcal{F}_t)$-martingale and a martingale with respect to its natural filtration.

Without loss of generality, we can assume that $B_0^2 = B_0^3 = 0$. Then

$$\mathsf{E}X_t \leq \mathsf{E}\frac{1}{\sqrt{(B_t^2)^2 + (B_t^3)^2}} = \frac{\text{const}}{\sqrt{t}} \xrightarrow[t\to\infty]{} 0.$$

This shows that X is not a martingale with respect to any filtration. □

The next example shows that the answer to the problem in formulations 3.B, 3.C, 7.B, and 7.C is negative.

Example 3.3. *Let B be a Brownian motion started at zero. For $n \in \mathbb{N}$, consider the function*

$$f^n(t) = k2^{-n} \text{ for } t \in [(k-1)2^{-n}, k2^{-n}), \quad k \in \mathbb{N},$$

define

$$\tau_1^n = \inf\{t \geq 0 : a_1^n B_t = f^n(t)\},$$
$$Y_t^n = a_1^n B_{t\wedge\tau_1^n}, \quad t \in [0, 2^{-n}),$$

and, for $k = 1, 2, \dots$, set

$$\tau_{k+1}^n = \inf\{t \geq k2^{-n} : Y_{k2^{-n}}^n + a_{k+1}^n(B_t - B_{k2^{-n}}) = f^n(t)\},$$
$$Y_t^n = Y_{k2^{-n}}^n + a_{k+1}^n\big(B_{t\wedge\tau_{k+1}^n} - B_{k2^{-n}}\big), \quad t \in [k2^{-n}, (k+1)2^{-n}),$$

where $(a_k^n)_{k\in\mathbb{N}}$ are positive real numbers growing to $+\infty$ so rapidly that

$$\mu_L\big(t \geq 0 : \mathsf{P}(Y_t^n = f^n(t)) \leq 1 - 2^{-n}\big) \leq 2^{-n} \tag{3.1}$$

(here μ_L denotes the Lebesgue measure). Let ξ be a random variable that is independent of B and has the exponential distribution with parameter 1. Set $X_t^n = Y_{\xi t}^n$, $X_t = \xi t$,

$\mathcal{G}_t = \sigma(\xi) \vee \mathcal{F}_t^B$, $\mathcal{F}_t = \mathcal{G}_{\xi t}$ *(note that, for any* $\alpha \geq 0$, $\xi\alpha$ *is a* $(\mathcal{G}_t)$*-stopping time). Then each* X^n *is a continuous local martingale with respect to its natural filtration as well as with respect to* $(\mathcal{F}_t)$*. Moreover,* $X_t^n \xrightarrow[n\to\infty]{\mathsf{P}} X_t$ *for any* $t \geq 0$ *(hence,* (X^n) *also converges to* X *in the sense of the weak convergence of finite-dimensional distributions). However,* X *is not a local martingale with respect to any filtration.*

`Proof.` Each process Y^n is a stochastic integral of a locally bounded $(\mathcal{F}_t^B)$-predictable process with respect to B. Hence, each Y^n is a continuous $(\mathcal{F}_t^B)$-local martingale. Consequently, each Y^n is a continuous $(\mathcal{G}_t)$-local martingale. This implies that each X^n is a continuous $(\mathcal{F}_t)$-local martingale (see [5; Ch. V, Prop. 1.5]). Due the continuity of X^n, each X^n is a local martingale with respect to its natural filtration.

It follows from (3.1) that $Y_t^n \xrightarrow[n\to\infty]{\text{a.s.}} t$ for μ_L-a.e. $t \geq 0$. Hence, $X_{\xi t}^n \xrightarrow[n\to\infty]{\text{a.s.}} \xi t$ for any $t \geq 0$.

The process X is not a local martingale with respect to any filtration since it has continuous paths of finite variation. □

The proposition below shows that the answer to the problem in formulations 4.B and 4.C is positive.

Proposition 3.4. *Let* (X^n) *be a sequence of local martingales (each with respect to its natural filtration) such that* $X_0^n = 0$ *and* $|\Delta X^n| \leq a$ *for some constant* $a > 0$*. Suppose that* (X^n) *converges in distribution to a process* X*. Then* X *is a local martingale (with respect to its natural filtration).*

For the proof, see [2; Ch. IX, Cor. 1.19].

The theorem below shows that the answer to the problem in formulations 8.B and 8.C is positive.

Theorem 3.5. *Let* (X^n) *be a sequence of* $(\mathcal{F}_t)$*-local martingales such that* $X_0^n = 0$ *and* $|\Delta X^n| \leq a$ *for some constant* $a > 0$*. Suppose that* (X^n) *converges in probability uniformly on compact intervals to a process* X*. Then* X *is an* $(\mathcal{F}_t)$*-local martingale.*

`Proof.` For $m, n \in \mathbb{N}$, set $\tau_m = \inf\{t : |X_t| \geq m\}$, $\sigma_{mn} = \inf\{t : |X_t^n| \geq 2m\}$. Then, for any $m \in \mathbb{N}$ and t, we have

$$\tau_m \wedge \sigma_{mn} \wedge t \xrightarrow[n\to\infty]{\mathsf{P}} \tau_m \wedge t,$$

and hence, the sequence of stopped processes $(X^n)^{\tau_m \wedge \sigma_{mn}}$ converges in probability uniformly on compact intervals as $n \to \infty$ to the stopped process X^{τ_m}. Note that

$$\big|(X^n)^{\tau_m \wedge \sigma_{mn}}\big| \leq 2m + a. \tag{3.2}$$

Hence, $(X^n)^{\tau_m \wedge \sigma_{mn}}$ is an $(\mathcal{F}_t)$-martingale, i.e. for any $s < t$, we have

$$\mathsf{E}\big((X^n)_t^{\tau_m \wedge \sigma_{mn}} \,\big|\, \mathcal{F}_s\big) = (X^n)_s^{\tau_m \wedge \sigma_{mn}}. \tag{3.3}$$

Combining the property

$$(X^n)_t^{\tau_m \wedge \sigma_{mn}} \xrightarrow[n\to\infty]{\mathsf{P}} X_t^{\tau_m}$$

with (3.2), we conclude that

$$(X^n)_t^{\tau_m \wedge \sigma_{mn}} \xrightarrow[n\to\infty]{L^1} X_t^{\tau_m}.$$

This, together with (3.3), shows that X^{τ_m} is an $(\mathcal{F}_t)$-martingale. As $\tau_m \xrightarrow[n\to\infty]{} \infty$, X is an $(\mathcal{F}_t)$-local martingale. □

The next theorem shows that the answer to the problem in formulations 1.D, 2.D, 3.D, and 4.D is positive.

Theorem 3.6. *Let (X^n) be a sequence of martingales (each with respect to its natural filtration) such that $|X^n| \le a$ for some constant $a > 0$. Suppose that (X^n) converges to a process X in the sense of the weak convergence of finite-dimensional distributions. Then X is a martingale (with respect to its natural filtration).*

Proof. Fix $s \le t$. For any $m \in \mathbb{N}$, any $s_1 \le \dots \le s_m \le s$, any bounded continuous function $f : \mathbb{R}^m \to \mathbb{R}$, and any $n \in \mathbb{N}$, we have

$$\mathsf{E}(X_t^n f(X_{s_1}^n, \dots, X_{s_m}^n)) = \mathsf{E}(X_s^n f(X_{s_1}^n, \dots, X_{s_m}^n)).$$

Letting $n \to \infty$, we get

$$\mathsf{E}(X_t f(X_{s_1}, \dots, X_{s_m})) = \mathsf{E}(X_s f(X_{s_1}, \dots, X_{s_m})).$$

By the Lebesgue dominated convergence theorem,

$$\mathsf{E}(X_t I(X_{s_1} \in A_1, \dots, X_{s_m} \in A_m)) = \mathsf{E}(X_s I(X_{s_1} \in A_1, \dots, X_{s_m} \in A_m))$$

for any intervals $A_1, \dots, A_m$. Due to the monotone class lemma,

$$\{C \in \mathcal{F}_s^X : \mathsf{E}(X_t I_C) = \mathsf{E}(X_s I_C)\} = \mathcal{F}_s^X.$$

This is the desired statement. □

The next theorem shows that the answer to the problem in formulations 5.D, 6.D, 7.D, and 8.D is positive.

Theorem 3.7. *Let (X^n) be a sequence of $(\mathcal{F}_t)$-martingales such that $|X^n| \le a$ for some constant $a > 0$. Suppose that $X_t^n \xrightarrow[n\to\infty]{\mathsf{P}} X_t$ for any t. Then X is an $(\mathcal{F}_t)$-martingale.*

Proof. For any $s \le t$ and any $n \in \mathbb{N}$, we have $\mathsf{E}(X_t^n \mid \mathcal{F}_s) = X_s$. Furthermore, $X_t^n \xrightarrow[n\to\infty]{L^1} X_t$. Hence, $\mathsf{E}(X_t \mid \mathcal{F}_s) = X_s$. □

4 Stochastic Integrals with Respect to a Martingale

It follows from [4; Cor. 21] that the answer to the problem "Is a stochastic integral of a bounded process with respect to a martingale also a martingale?" is negative. Here we give an explicit counter-example (it follows from [4] that such an example exists, but it is not constructed explicitly).

We construct a uniformly integrable $(\mathcal{F}_t)$-martingale $X = (X_t)_{t\ge0}$ and a bounded $(\mathcal{F}_t)$-predictable process $H = (H_t)_{t\ge0}$ such that the stochastic integral of H with respect to X is not a uniformly integrable martingale. This yields the negative answer to the problem under consideration. Indeed, the process

$$\widetilde{X}_t = \begin{cases} X_{\frac{t}{1-t}}, & t < 1, \\ X_\infty, & t \ge 1 \end{cases}$$

Formulation \ Additional assumptions	**A.** No additional assumptions	**B.** $X_0^n = 0$ and $\lvert\Delta X^n\rvert \le a$	**C.** $X_0^n = 0$ and X^n are continuous	**D.** $\lvert X^n\rvert \le a$
1. $X^n \in \mathcal{M}$, $X^n \xrightarrow[n\to\infty]{\text{FD}} X \overset{?}{\Longrightarrow} X \in \mathcal{M}$	No, Example 3.1	No, Example 3.2	No, Example 3.2	Yes, Theorem 3.6
2. $X^n \in \mathcal{M}$, $X^n \xrightarrow[n\to\infty]{\text{Law}} X \overset{?}{\Longrightarrow} X \in \mathcal{M}$	No, Example 3.1	No, Example 3.2	No, Example 3.2	Yes, Theorem 3.6
3. $X^n \in \mathcal{M}_{\text{loc}}$, $X^n \xrightarrow[n\to\infty]{\text{FD}} X \overset{?}{\Longrightarrow} X \in \mathcal{M}_{\text{loc}}$	No, Example 3.1	No, Example 3.3	No, Example 3.3	Yes, Theorem 3.6
4. $X^n \in \mathcal{M}_{\text{loc}}$, $X^n \xrightarrow[n\to\infty]{\text{Law}} X \overset{?}{\Longrightarrow} X \in \mathcal{M}_{\text{loc}}$	No, Example 3.1	Yes, Prop. 3.4	Yes, Prop. 3.4	Yes, Theorem 3.6
5. $X^n \in \mathcal{M}(\mathcal{F}_t)$, $\forall t\ X_t^n \xrightarrow[n\to\infty]{\mathsf{P}} X_t \overset{?}{\Longrightarrow} X \in \mathcal{M}(\mathcal{F}_t)$	No, Example 3.1	No, Example 3.2	No, Example 3.2	Yes, Theorem 3.7
6. $X^n \in \mathcal{M}(\mathcal{F}_t)$, $X^n \xrightarrow[n\to\infty]{\text{u.p.}} X \overset{?}{\Longrightarrow} X \in \mathcal{M}(\mathcal{F}_t)$	No, Example 3.1	No, Example 3.2	No, Example 3.2	Yes, Theorem 3.7
7. $X^n \in \mathcal{M}_{\text{loc}}(\mathcal{F}_t)$, $\forall t\ X_t^n \xrightarrow[n\to\infty]{\mathsf{P}} X_t \overset{?}{\Longrightarrow} X \in \mathcal{M}_{\text{loc}}(\mathcal{F}_t)$	No, Example 3.1	No, Example 3.3	No, Example 3.3	Yes, Theorem 3.7
8. $X^n \in \mathcal{M}_{\text{loc}}(\mathcal{F}_t)$, $X^n \xrightarrow[n\to\infty]{\text{u.p.}} X \overset{?}{\Longrightarrow} X \in \mathcal{M}_{\text{loc}}(\mathcal{F}_t)$	No, Example 3.1	Yes, Theorem 3.5	Yes, Theorem 3.5	Yes, Theorem 3.7

Table 2. Summary of the answers to the problem "Is a limit of a converging sequence of martingales also a martingale?". Here we use the notation from Table 1 and the additional notation: "$X^n \xrightarrow[n\to\infty]{\text{FD}} X$" means that (X^n) converges to X in the sense of the weak convergence of finite-dimensional distributions; "$X^n \xrightarrow[n\to\infty]{\text{Law}} X$" means that (X^n) converges to X in distribution; "$X^n \xrightarrow[n\to\infty]{\text{u.p.}} X$" means that (X^n) converges to X in probability uniformly on compact intervals.

is a martingale with respect to the filtration

$$\widetilde{\mathcal{F}}_t = \begin{cases} \mathcal{F}_{\frac{t}{1-t}}, & t < 1, \\ \mathcal{F}, & t \geq 1. \end{cases}$$

Furthermore, the stochastic integral of the process $\widetilde{H}_t = H_{\frac{t}{1-t}} I(t < 1)$ with respect to $\widetilde{X}$ is not a martingale in view of the equality

$$\int_0^t \widetilde{H}_s d\widetilde{X}_s = \int_0^{\frac{t}{1-t}} H_s dX_s, \quad t < 1.$$

Example 4.1. *Let*

$$a_n = 2n, \quad b_n = \frac{2n}{2n^2 - n + 1}, \quad p_n = \frac{n-1}{2n^2}, \quad n \in \mathbb{N}.$$

Construct the sequence $(X_n)_{n\in\mathbb{N}}$ and the sequence of sets $(A_n)_{n\in\mathbb{N}}$ by

$$\begin{aligned} &X_0 = 1, \quad X_1 = 1, \quad A_1 = \Omega, \ldots \\ &\mathsf{P}(X_{n+1} = a_2 \ldots a_{n+1} \mid A_n) = p_{n+1}, \\ &\mathsf{P}(X_{n+1} = a_2 \ldots a_n b_{n+1} \mid A_n) = 1 - p_{n+1}, \\ &\mathsf{P}(X_{n+1} = X_n \mid A_n^c) = 1, \\ &A_{n+1} = \{X_{n+1} = a_1 \ldots a_{n+1}\}, \ldots \end{aligned}$$

Define the continuous-time process $(X_t)_{t\geq 0}$ by $X_t = X_n$ for $t \in [n, n+1)$. Set $\mathcal{F}_t = \mathcal{F}_t^X$ and consider

$$H_t = \sum_{n=1}^{\infty} I(2n - 1 < t \leq 2n).$$

Then X is a uniformly integrable $(\mathcal{F}_t)$-martingale, while the stochastic integral of H with respect to X is not a uniformly integrable $(\mathcal{F}_t)$-martingale.

`Proof`. Clearly, X is an $(\mathcal{F}_t)$-martingale. For any $n < m \in \mathbb{N}$, we have

$$\begin{aligned} &\mathsf{E}|(X_m - X_n)| = \mathsf{E}|(X_m - X_n) I_{A_n}| \\ &= \mathsf{E}|(X_m - X_n) I_{A_{n+1}}| + \mathsf{E}|(X_m - X_n) I_{A_n} I_{A_{n+1}^c}| \\ &= \mathsf{E}|(X_m - X_n) I_{A_{n+1}}| + \mathsf{E}|(X_{n+1} - X_n) I_{A_n} I_{A_{n+1}^c}|. \end{aligned}$$

One can check by the induction in m that $(X_m - X_n) I_{A_{n+1}} > 0$ for $m > n$. Thus,

$$\mathsf{E}|(X_m - X_n) I_{A_{n+1}}| = \mathsf{E}(X_m - X_n) I_{A_{n+1}} = \mathsf{E}(X_{n+1} - X_n) I_{A_{n+1}} = \mathsf{E}|(X_{n+1} - X_n) I_{A_{n+1}}|,$$

and consequently,

$$\begin{aligned} &\mathsf{E}|(X_m - X_n)| = \mathsf{E}|(X_{n+1} - X_n) I_{A_n}| \\ &= a_2 \ldots a_n (a_{n+1} - 1) p_2 \ldots p_n p_{n+1} + a_2 \ldots a_n (1 - b_{n+1}) p_2 \ldots p_n (1 - p_{n+1}) \\ &\leq a_2 \ldots a_n p_2 \ldots p_n (a_{n+1} p_{n+1} + 1) = \frac{1}{n} \left(\frac{n}{n+1} + 1 \right) \leq \frac{2}{n}. \end{aligned}$$

As a result, the sequence $(X_n)_{n\in\mathbb{N}}$ converges in L^1, which means that X is a uniformly integrable $(\mathcal{F}_t)$-martingale.

Furthermore, for any $n \le m \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{E}\left|I_{A_{2n}} I_{A^c_{2n+1}} \int_0^{2m} H_s dX_s\right| &= \mathsf{E}\Big(I_{A_{2n}} I_{A^c_{2n+1}} \sum_{k=1}^{n} (X_{2k} - X_{2k-1})\Big) \\
&\ge \mathsf{E}\Big(I_{A_{2n}} I_{A^c_{2n+1}} (X_{2n} - X_{2n-1})\Big) \\
&= p_2 \dots p_{2n}(1 - p_{2n+1}) a_2 \dots a_{2n-1}(a_{2n} - 1) \\
&\ge \frac{1}{4} p_2 \dots p_{2n} a_2 \dots a_{2n} = \frac{1}{8n}.
\end{aligned}$$

Therefore,

$$\mathsf{E}\left|\int_0^{2m} H_s dX_s\right| \ge \sum_{n=1}^{m} \mathsf{E}\left|I_{A_{2n}} I_{A^c_{2n+1}} \int_0^{2m} H_s dX_s\right| \ge \sum_{n=1}^{m} \frac{1}{8n} \xrightarrow[m\to\infty]{} \infty.$$

As a result, the stochastic integral of H with respect to X is not uniformly integrable. □

5 Uniform Integrability of Martingales

The answer to the problem "If $X = (X_t)_{t\ge0}$ is a positive process such that $\mathsf{E}X_\tau = \mathsf{E}X_0 < \infty$ for any finite stopping time τ, then is it true that X is a uniformly integrable martingale?" is positive as shown by the following theorem.

Theorem 5.1. *Let $(\mathcal{F}_t)$ be a filtration satisfying the usual assumptions of right-continuity and completeness. Let $X = (X_t)_{t\ge0}$ be an $(\mathcal{F}_t)$-adapted positive càdlàg process such that $\mathsf{E}X_\tau = \mathsf{E}X_0 < \infty$ for any $(\mathcal{F}_t)$-stopping time τ that is finite a.s. Then X is a uniformly integrable $(\mathcal{F}_t)$-martingale.*

`Proof`. Fix $s \le t$ and $A \in \mathcal{F}_s$. Consider stopping times $\tau_1 = s$ and $\tau_2 = sI_{A^c} + tI_A$. Then the equality $\mathsf{E}X_{\tau_1} = \mathsf{E}X_{\tau_2}$ implies that $\mathsf{E}X_t I_A = \mathsf{E}X_s I_A$. As a result, X is an $(\mathcal{F}_t)$-martingale.

Since X is positive, there exists a limit $X_\infty = \text{(a.s.)} \lim_{t\to\infty} X_t$. By the Fatou lemma for conditional expectations,

$$\mathsf{E}(X_\infty \mid \mathcal{F}_t) \le X_t, \quad t \ge 0. \tag{5.1}$$

In particular, $\mathsf{E}X_\infty \le \mathsf{E}X_0$.

Suppose that $\mathsf{E}X_\infty < \mathsf{E}X_0$. The process $\widetilde{X}_t = \mathsf{E}(X_\infty \mid \mathcal{F}_t)$, $t \ge 0$ has a càdlàg modification. Moreover, $\widetilde{X}_t \xrightarrow[t\to\infty]{\text{a.s.}} X_\infty$. Consequently, the stopping time

$$\tau = \inf\left\{t \ge 0 : |X_t - \widetilde{X}_t| \le \frac{\mathsf{E}X_0 - \mathsf{E}X_\infty}{2}\right\}$$

is finite a.s. By the conditions of the theorem, $\mathsf{E}X_\tau = \mathsf{E}X_0$, which implies that

$$\mathsf{E}\widetilde{X}_\tau > \mathsf{E}X_0 - \frac{\mathsf{E}X_0 - \mathsf{E}X_\infty}{2} > \mathsf{E}X_\infty.$$

This contradicts the equality $\mathsf{E}\widetilde{X}_\tau = \mathsf{E}X_\infty$, which is a consequence of the optional stopping theorem for uniformly integrable martingales. As a result, $\mathsf{E}X_\infty = \mathsf{E}X_0$. This, combined with (5.1), shows that $\mathsf{E}(X_\infty \mid \mathcal{F}_t) = X_t$ for any $t \geq 0$. The proof is completed. □

We conclude the paper by the following

Question. *Let $X = (X_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted càdlàg process such that, for any $(\mathcal{F}_t)$-stopping time τ that is finite a.s., the random variable X_τ is integrable and $\mathsf{E}X_\tau = \mathsf{E}X_0$. Is it true that X is a uniformly integrable $(\mathcal{F}_t)$-martingale?*

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