SOME DISTRIBUTIONAL PROPERTIES OF THE BROWNIAN MOTION WITH A DRIFT AND AN EXTENSION OF P. LÉVY'S THEOREM

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Abstract. The theorem proved by P. Lévy states that $\big(\sup B - B, \sup B\big) \overset{\text{law}}{=} \big(|B|, L(B)\big)$. Here, B is the standard linear Brownian motion and $L(B)$ is its local time in zero. In this paper, we present an extension of P. Lévy's theorem to the case of the Brownian motion with a (random) drift as well as to the case of conditionally Gaussian martingales. Besides, we give a simple proof of the equality $2\sup B^\lambda - B^\lambda \overset{\text{law}}{=} |B^\lambda| + L(B^\lambda)$ where B^λ is the Brownian motion with drift $\lambda \in \mathbb{R}$.

Key words and phrases. P. Lévy's theorem, local time, Brownian motion with a drift, conditionally Gaussian martingales, Skorokhod's lemma.

1 An Invariance Property of the Brownian Motion

1. Let $B = (B_t)_{t\ge0}$ be the standard linear Brownian motion and $B^\lambda = (B^\lambda_t)_{t\ge0}$ be the Brownian motion with a drift $(B^\lambda_t = \lambda t + B_t)$ where $\lambda \in \mathbb{R}$.

The classical theorem proved by P. Lévy (see [3], [9, ch.VI, §2, (2.3)]) states that

$$\big(\sup B - B, \sup B\big) \overset{\text{law}}{=} \big(|B|, L(B)\big), \tag{1}$$

i.e., the processes $\big(\sup_{s\le t} B_s - B_t, \sup_{s\le t} B_s;\ t\ge 0\big)$ and $\big(|B_t|, L_t(B);\ t\ge0\big)$ have the same law. Here, $L(B) = (L_t(B))_{t\ge0}$ is the local time of B in zero:

$$L_t(B) = \lim_{\varepsilon\downarrow0} \frac{1}{2\varepsilon}\int_0^t I\big(|B_s| \le \varepsilon\big)\, ds \quad \text{a.s.}$$

It follows from (1) that

$$2\sup B - B \overset{\text{law}}{=} |B| + L(B). \tag{2}$$

This "one-dimensional" property was extended in [10] to the case of the Brownian motion with a drift:

$$2\sup B^\lambda - B^\lambda \overset{\text{law}}{=} |B^\lambda| + L(B^\lambda). \tag{3}$$

The paper [1] presents an extension of the "two-dimensional" P. Lévy's result. Namely, it is proved in [1] that

$$\big(\sup B^\lambda - B^\lambda,\, \sup B^\lambda\big) \overset{\text{law}}{=} \big(|X^\lambda|,\, L(X^\lambda)\big). \tag{4}$$

Here, $X^\lambda = (X_t^\lambda)_{t\ge 0}$ is the (strong) solution of the stochastic differential equation (SDE)

$$dX_t^\lambda = -\lambda \operatorname{sgn} X_t^\lambda\, dt + dB_t, \quad X_0^\lambda = 0 \tag{5}$$

and $L(X^\lambda) = (L_t(X^\lambda))_{t\ge 0}$ is the local time of X^λ in zero.

It follows from (4) that

$$2\sup B^\lambda - B^\lambda \overset{\text{law}}{=} |X^\lambda| + L(X^\lambda). \tag{6}$$

Equalities (3) and (6) taken together yield

Theorem 1. *For any* $\lambda \in \mathbb{R}$,

$$|B^\lambda| + L(B^\lambda) \overset{\text{law}}{=} |X^\lambda| + L(X^\lambda). \tag{7}$$

It makes sense to give a straightforward proof of this invariance in distribution for $|x| + L(x)$, $x \in C(\mathbb{R}_+, \mathbb{R})$ when $x = B^\lambda$ is replaced by $x = X^\lambda$ since this property is of some interest by itself. Note that $(6) + (7) \Rightarrow (3)$ and $(3) + (7) \Rightarrow (6)$.

Proof of Theorem 1. It follows from P. Lévy's theorem (1) that for any $\lambda \in \mathbb{R}$ and $t \ge 0$,

$$\mathsf{E}\, e^{\lambda B_t} = \mathsf{E}\, e^{\lambda(L_t(B) - |B_t|)}. \tag{8}$$

The following "conditional" version of equality (8) is the key point in the proof of Theorem 1: for any $t \ge 0$,

$$\mathsf{E}\Big[e^{\lambda B_t}\Big|\mathcal{F}_t^{R(B)}\Big] = \mathsf{E}\Big[e^{\lambda(L_t(B) - |B_t|)}\Big|\mathcal{F}_t^{R(B)}\Big] \quad \text{a.s.} \tag{9}$$

Here, $R(B) = (R_t(B))_{t\ge 0}$, $R_t(B) = |B_t| + L_t(B)$ and $\mathcal{F}_t^{R(B)} = \sigma(R_s(B);\, s \le t)$. Furthermore,

$$\mathsf{E}\Big[e^{\lambda B_t}\Big|\mathcal{F}_t^{R(B)}\Big] = \frac{\operatorname{sh} \lambda R_t(B)}{\lambda R_t(B)}. \tag{10}$$

In order to prove (9) and (10), we first note that the symmetry considerations lead to the equality

$$\mathsf{E}\Big[e^{\lambda B_t}\Big|\mathcal{F}_t^{|B|}\Big] = \frac{1}{2}\Big(e^{\lambda|B_t|} + e^{-\lambda|B_t|}\Big) \ \Big(= \operatorname{ch} \lambda|B_t|\Big)$$

where $\mathcal{F}_t^{|B|} = \sigma(|B_s|;\, s \le t)$. This, combined with the inclusion $\mathcal{F}_t^{R(B)} \subseteq \mathcal{F}_t^{|B|}$, implies that

$$\mathsf{E}\Big[e^{\lambda B_t}\Big|\mathcal{F}_t^{R(B)}\Big] = \frac{1}{2}\mathsf{E}\Big[e^{\lambda|B_t|} + e^{-\lambda|B_t|}\Big|\mathcal{F}_t^{R(B)}\Big]. \tag{11}$$

It follows from [8] (see also [9, ch.VI, §3, (3.6)]) that the conditional distribution $\operatorname{Law}\big(|B_t|\big|\mathcal{F}_t^{R(B)}\big)$ is the uniform distribution on $[0, R_t(B)]$. Therefore,

$$\frac{1}{2}\mathsf{E}\Big[e^{\lambda|B_t|} + e^{-\lambda|B_t|}\Big|\mathcal{F}_t^{R(B)}\Big] = \frac{1}{2\lambda R_t(B)}\Big(e^{\lambda R_t(B)} - e^{-\lambda R_t(B)}\Big).$$

This, together with (11), proves (10).

Similarly, we deduce that

$$\mathsf{E}\Big[e^{\lambda(L_t(B)-|B_t|)}\Big|\mathcal{F}_t^{R(B)}\Big] = e^{\lambda R_t(B)}\,\mathsf{E}\Big[e^{-2\lambda|B_t|}\Big|\mathcal{F}_t^{R(B)}\Big] =$$

$$= \frac{1}{2\lambda R_t(B)}\Big(e^{\lambda R_t(B)} - e^{-\lambda R_t(B)}\Big) = \frac{\operatorname{sh}\lambda R_t(B)}{\lambda R_t(B)}. \quad (12)$$

Combining (10) and (12), we obtain the desired equality (9).

We now turn to the proof of the invariance property (7). Let P_B, P_{B^λ} and P_{X^λ} be the distributions of the processes B, B^λ and X^λ on the canonical path space $C(\mathbb{R}_+,\mathbb{R})$. We will use $P_B|\mathcal{F}_t$, $P_{B^\lambda}|\mathcal{F}_t$ and $P_{X^\lambda}|\mathcal{F}_t$ to denote the restrictions of these measures to the σ-fields $\mathcal{F}_t=\sigma(x_s;\, s\le t)$, $t\ge 0$ where $(x_t)_{t\ge 0}$ is the canonical process on $C(\mathbb{R}_+,\mathbb{R})$.

It is well known (see, for example, [4, ch.7, §2]) that

$$\frac{d(P_{B^\lambda}|\mathcal{F}_t)}{d(P_B|\mathcal{F}_t)}(B) = e^{\lambda B_t - \frac{\lambda^2}{2}t}. \quad (13)$$

Taking into account *Tanaka's formula*,

$$|B_t| = \int_0^t \operatorname{sgn} B_s\, dB_s + L_t(B)$$

(see [9, ch.VI, §1]), we obtain

$$\frac{d(P_{X^\lambda}|\mathcal{F}_t)}{d(P_B|\mathcal{F}_t)}(B) = e^{-\lambda\int_0^t \operatorname{sgn} B_s\, dB_s - \frac{\lambda^2}{2}t} = e^{\lambda(L_t(B)-|B_t|)-\frac{\lambda^2}{2}t}. \quad (14)$$

We will use $P_{R(B)}$, $P_{R(B^\lambda)}$ and $P_{R(X^\lambda)}$ to denote the images of P_B, P_{B^λ} and P_{X^λ} under the map

$$C(\mathbb{R}_+,\mathbb{R}) \ni x \longmapsto R(x) \in C(\mathbb{R}_+,\mathbb{R})$$

where $R_t(x) = |x_t| + L_t(x)$. Set $\mathcal{F}_t^R = \sigma(R_s(x);\, s\le t)$. It can easily be checked that (P_B – a.s.)

$$\frac{d\big(P_{R(B^\lambda)}|\mathcal{F}_t^R\big)}{d\big(P_{R(B)}|\mathcal{F}_t^R\big)}(R(B)) = \mathsf{E}\left[\frac{d\big(P_{B^\lambda}|\mathcal{F}_t\big)}{d\big(P_B|\mathcal{F}_t\big)}(B)\middle|\mathcal{F}_t^{R(B)}\right] \quad (15)$$

and

$$\frac{d\big(P_{R(X^\lambda)}|\mathcal{F}_t^R\big)}{d\big(P_{R(B)}|\mathcal{F}_t^R\big)}(R(B)) = \mathsf{E}\left[\frac{d\big(P_{X^\lambda}|\mathcal{F}_t\big)}{d\big(P_B|\mathcal{F}_t\big)}(B)\middle|\mathcal{F}_t^{R(B)}\right]. \quad (16)$$

Combining (13), (14) and (9), we see that the right-hand sides in (15) and (16) coincide (P_B – a.s.) and

$$\frac{d\big(P_{R(B^\lambda)}|\mathcal{F}_t^R\big)}{d\big(P_{R(B)}|\mathcal{F}_t^R\big)}(R(B)) = \frac{d\big(P_{R(X^\lambda)}|\mathcal{F}_t^R\big)}{d\big(P_{R(B)}|\mathcal{F}_t^R\big)}(R(B)) = e^{-\frac{\lambda^2}{2}t}\,\frac{\operatorname{sh}\lambda R_t(B)}{\lambda R_t(B)}. \quad (17)$$

Therefore, $P_{R(B^\lambda)} = P_{R(X^\lambda)}$. This completes the proof of Theorem 1.

2. R e m a r k s. 1) It follows from (17) that $R(B^\lambda) \overset{\text{law}}{=} R(B^{-\lambda})$ and $R(X^\lambda) \overset{\text{law}}{=} R(X^{-\lambda})$.

2) The results of [8] (see also [9, ch.VI, §3, (3.5)]) imply that the process $2\sup B - B$ has the same law as the *three-dimensional Bessel process* $\operatorname{Bes}(3)$ started at zero. The infinitesimal generator $\mathcal{A}_\nu$ of the ν-dimensional Bessel process $\operatorname{Bes}(\nu)$ is given by

$$\mathcal{A}_\nu = \frac{\nu-1}{2r}\frac{d}{dr} + \frac{1}{2}\frac{d^2}{dr^2}.$$

Thus, for $\nu = 3$, we have

$$\mathcal{A}_3 = \frac{d}{dr} + \frac{1}{2}\frac{d^2}{dr^2}.$$

This implies that the process Bes(3) is a (strong) solution of the SDE

$$dr_t = \frac{1}{r_t}dt + d\beta_t, \quad r_0 = 0$$

where $\beta = (\beta_t)_{t\geq 0}$ is the Brownian motion.

It follows from (2) that the process $R(B) = |B| + L(B)$ is also a version of the Bessel process Bes(3).

3) It was proved in [10] that the process $2\sup B^\lambda - B^\lambda$ is a diffusion process with the following infinetisemal generator:

$$\mathcal{A}_3^\lambda = \lambda \coth \lambda r \frac{d}{dr} + \frac{1}{2}\frac{d^2}{dr^2}.$$

Equalities (3) and (6) imply that each of the processes $|B^\lambda| + L(B^\lambda)$ and $|X^\lambda| + L(X^\lambda)$ is a diffusion process with infinitesimal generator $\mathcal{A}_3^\lambda$.

4) Due to equalities (15) and (17), the process

$$M_t = e^{-\frac{\lambda^2}{2}t}\,\frac{\operatorname{sh}\lambda R_t(B)}{\lambda R_t(B)}$$

is a $\big(\mathcal{F}_t^{R(B)}, P_{R(B)}\big)$-martingale. Set $\tau_a = \inf\{t \geq 0 : R_t(B) = a\}$, $a > 0$. It is clear that $P(\tau_a < \infty) = 1$ and the family of random variables $\{M_{t\wedge\tau_a}, t \geq 0\}$ is uniformly integrable. Applying the *optional stopping theorem*, we get $\mathsf{E}M_{\tau_a} = 1$. This is equivalent to the well-known property

$$\mathsf{E}\, e^{-\frac{\lambda^2}{2}\tau_a} = \frac{\lambda a}{\operatorname{sh}\lambda a}.$$

By comparison, for $\sigma_a = \inf\{t : |B_t| \geq a\}$, one has

$$\mathsf{E}\, e^{-\frac{\lambda^2}{2}\sigma_a} = \frac{1}{\operatorname{ch}\lambda a}.$$

It is obvious that $\tau_a \leq \sigma_a$.

2 Some Extensions of P. Lévy's Theorem

1. The proof of property (4) was given in [1]. This proof is based on the application of *Girsanov's theorem* to the processes B^λ and X^λ, *Tanaka's formula* to $|B|$ and *P. Lévy's theorem* (1) to the process B. However, it turns out that property (4) can be deduced directly from *Skorokhod's lemma* which is formulated below. This lemma is usually used to prove P. Lévy's theorem (see, for example, [9, ch.VI, §2, (2.1)]). Moreover, Skorokhod's lemma makes it possible to give new extensions of property (1) (see Theorems 3 and 4 below).

Lemma (Skorokhod). *Let $y = (y_t)_{t\geq 0}$ be a continuous function such that $y_0 \geq 0$. There exists a unique pair of functions $(x, l) = (x_t, l_t)_{t\geq 0}$ such that*

(a) $x = y + l$,

(b) $x \geq 0$,

(c) $l = (l_t)_{t\geq 0}$ *is increasing, continuous, vanishing at zero, and the corresponding measure dl_t is carried by $\{s \geq 0 : x_s = 0\}$.*

The function l is moreover given by

$$l_t = \sup_{s\leq t}(-y_s \vee 0).$$

For the proof of this lemma, see [11], [9, ch.VI, §2, (2.1)].

2. We now turn to the extension of P. Lévy's theorem to the Brownian motion with a *random* drift.

First, let us consider the SDEs

$$dB_t^{(a)} = a\big(t, B^{(a)}\big)\,dt + dB_t, \quad B_0^{(a)} = 0 \tag{18}$$

and

$$dX_t^{(a)} = -a\big(t, Y^{(a)}\big)\ \mathrm{sgn}\, X_t^{(a)}\,dt + dB_t, \quad X_0^{(a)} = 0. \tag{19}$$

Here, $a = a(t,x)$ is a bounded process on $C(\mathbb{R}_+, \mathbb{R})$ that is predictable with respect to the canonical filtration $\mathcal{F}_t = \sigma(x_s,\ s \le t)$, and

$$Y_t^{(a)} = -\int_0^t \mathrm{sgn}\, X_s^{(a)}\,dX_s^{(a)}. \tag{20}$$

For a constant drift $a(t,x) \equiv \lambda$, the solution of (18) is the Brownian motion with drift λ while SDE (19) transforms to (5).

Lemma 2. *There is weak existence and uniqueness in law for SDE (18) as well as for system (19)-(20).*

P r o o f. First, we prove the existence of a weak solution for system (19)-(20). Consider the canonical path space $C(\mathbb{R}_+, \mathbb{R}^2)$ with the canonical process $(x,y) = (x_t, y_t)_{t\ge 0}$ and the filtration $\mathcal{F}_t = \sigma(x_s, y_s;\ s \le t)$. Let P be a probability measure on $\mathcal{F}_\infty = \sigma(x_s, y_s;\ s \ge 0)$ such that $(x_t)_{t\ge 0}$ is the Brownian motion with respect to P and

$$y_t = -(P)\!\int_0^t \mathrm{sgn}\, x_s\,dx_s$$

where the symbol (P) denotes that the stochastic integral is constructed with respect to P.

Let us define the measures P'_t on the σ-fields $\mathcal{F}_t\ (t \ge 0)$ by

$$\frac{dP'_t}{dP_t}(x,y) = M_t = \exp\Big\{-(P)\!\int_0^t a(s,y)\,dx_s - \frac{1}{2}\int_0^t a^2(s,y)\,ds\Big\} \tag{21}$$

where $P_t = P|\mathcal{F}_t$. In view of the boundedness of $a(s,y)$, *Novikov's criterion* (see [6], [4, ch.6, §2]) implies that M_t is a $(\mathcal{F}_t, P)$-martingale. Thus, $P'_t|\mathcal{F}_s = P'_s$ for $s \le t$. Therefore, there exists a probability measure P' on $\mathcal{F}_\infty$ such that $P'|\mathcal{F}_t = P'_t$ for any $t \ge 0$ (see [12, p. 34]).

As the stochastic integral remains unchanged with the (locally) equivalent change of measure (see [5, Lemme III.2.]), we have

$$y_t = -(P)\!\int_0^t \mathrm{sgn}\, x_s\,dx_s = -(P')\!\int_0^t \mathrm{sgn}\, x_s\,dx_s.$$

This equality, combined with *Girsanov's theorem* (see [2, ch.III, §3b, (3.11)]), proves that the process $(x_t, y_t)_{t\ge 0}$ is a solution of system (19)-(20) on the filtered probability space $\big(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{F}_t, P'\big)$.

We now prove the uniqueness in law for system (19)-(20). Let Q' be a probability measure on $\big(C(\mathbb{R}_+, \mathbb{R}^2),\ \mathcal{F}_\infty\big)$ corresponding to an arbitrary weak solution of this system. Define the measures Q_t on the σ-fields $\mathcal{F}_t\ (t \ge 0)$ by

$$\frac{dQ_t}{dQ'_t}(x,y) = \exp\Big\{(Q')\!\int_0^t a(s,y)\,dx_s + \frac{1}{2}\int_0^t a^2(s,y)\,ds\Big\}. \tag{22}$$

Arguing as above, we see that there exists a probability measure Q on $\mathcal{F}_\infty$ such that $Q|\mathcal{F}_t = Q_t$ for any $t \geq 0$. By Girsanov's theorem, the process $(x_t)_{t\geq 0}$ is $(\mathcal{F}_t, P)$-Brownian motion. Furthermore,

$$y_t = -(Q')\int_0^t \operatorname{sgn} x_s \, dx_s = -(Q)\int_0^t \operatorname{sgn} x_s \, dx_s.$$

Thus, $Q = P$. With the equality

$$(Q')\int_0^t a(s,y)\, dx_s = (Q)\int_0^t a(s,y)\, dx_s = (P)\int_0^t a(s,y)\, dx_s,$$

formulas (21) and (22) imply that

$$\frac{dP'_t}{dP_t} = \left(\frac{dQ_t}{dQ'_t}\right)^{-1} = \frac{dQ'_t}{dQ_t}, \qquad t \geq 0.$$

Hence, $Q' = P'$. This proves the uniqueness in law for system (19)-(20).

The weak existence and the uniqueness in law for SDE (18) are proved in the same way.

Theorem 3. *Let $B^{(a)}$ and $X^{(a)}$ be solutions of SDEs (18)–(20). Then,*

$$\big(\sup B^{(a)} - B^{(a)},\, \sup B^{(a)}\big) \overset{\text{law}}{=} \big(|X^{(a)}|,\, L(X^{(a)})\big), \tag{23}$$

i.e., the processes $\big(\sup_{s\leq t} B_s^{(a)} - B_t^{(a)},\, \sup_{s\leq t} B_s^{(a)};\, t\geq 0\big)$ and $\big(|X^{(a)}|,\, L(X^{(a)});\, t\geq 0\big)$ have the same law.

P r o o f. By Tanaka's formula,

$$|X_t^{(a)}| = \int_0^t \operatorname{sgn} X_s^{(a)}\, dX_s^{(a)} + L_t(X^{(a)}) = -Y_t^{(a)} + L_t(X^{(a)}).$$

Applying Skorokhod's lemma, we derive that $L_t(X^{(a)}) = \sup_{s\leq t} Y_s^{(a)}$. Thus,

$$|X_t^{(a)}| = \sup_{s\leq t} Y_s^{(a)} - Y_t^{(a)}$$

and obviously,

$$\big(\sup_{s\leq t} Y_s^{(a)} - Y_t^{(a)},\, \sup_{s\leq t} Y_t^{(a)};\, t\geq 0\big) = \big(|X_t^{(a)}|,\, L_t(X^{(a)});\, t\geq 0\big). \tag{24}$$

Further,

$$Y_t^{(a)} = -\int_0^t \operatorname{sgn} X_s^{(a)}\, dX_s^{(a)} = \int_0^t a\big(s, Y^{(a)}\big)\, ds - \int_0^t \operatorname{sgn} X_s^{(a)}\, dB_s = \\ = \int_0^t a\big(s, Y^{(a)}\big)\, ds + \widetilde{B}_t. \tag{25}$$

It follows from P. Lévy's theorem (see [9, ch.IV, §3, (3.6)]) that the process $\widetilde{B}_t = -\int_0^t \operatorname{sgn} X_s^{(a)}\, dB_s$ is the standard linear Brownian motion. It is obvious that $\widetilde{B}$ is a martingale with respect to the filtration $\mathcal{F}^{Y^{(a)}} = (\mathcal{F}_t^{Y^{(a)}})_{t\geq 0}$. Thus, $Y^{(a)}$ is a solution of SDE (18). The uniqueness in law for this SDE implies that $Y^{(a)} \overset{\text{law}}{=} B^{(a)}$. Now, the desired result follows from (24).

3. In all the cases above, the "basic" process is expressed by the Brownian motion $B = (B_t)_{t\geq 0}$. In what follows, the "generating" process will be a *conditionally Gaussian* (with respect to the filtration $\mathcal{F}^{\langle M\rangle} = (\mathcal{F}_t^{\langle M\rangle})_{t\geq 0}$) martingale $M = B_{\langle M\rangle} = (B_{\langle M\rangle_t})_{t\geq 0}$ for which B and the quadratic variation $\langle M\rangle$ are *independent* processes (see [2, ch.II, (6.2)], [7]). In [13] such martingales are called *Ocone martingales.*

Theorem 4. *Let $M = B_{\langle M\rangle}$ be a conditionally Gaussian martingale with independent processes B and $\langle M\rangle$. Set*

$$M_t^\lambda = \lambda\langle M\rangle_t + M_t.$$

Let X^λ denote a solution of the SDE

$$dX_t^\lambda = -\lambda \operatorname{sgn} X_t^\lambda \, d\langle M\rangle_t + dM_t, \quad X_0^\lambda = 0. \tag{26}$$

Then,

$$\big(\sup M^\lambda - M^\lambda, \sup M^\lambda\big) \overset{\text{law}}{=} \big(|X^{(a)}|, L(X^{(a)})\big). \tag{27}$$

P r o o f. We first prove that SDE (26) has a solution. Let $B = (B_t)_{t\geq 0}$ be the initial Brownian motion and $Z^\lambda = (Z_t^\lambda)_{t\geq 0}$ be the solution of the SDE

$$dZ_t^\lambda = -\lambda \operatorname{sgn} Z_t^\lambda \, dt + dB_t, \quad Z_0^\lambda = 0.$$

This equation (compare with (5)) has the unique strong solution (see [14]). Due to the properties of the time-change $(t \mapsto \langle M\rangle_t)$, we have

$$Z_{\langle M\rangle_t}^\lambda = -\lambda \int_0^{\langle M\rangle_t} \operatorname{sgn} Z_s^\lambda \, ds + B_{\langle M\rangle_t} = -\lambda \int_0^t \operatorname{sgn} Z_{\langle M\rangle_s}^\lambda \, d\langle M\rangle_s + M_t.$$

Thus, the process $X_t^\lambda = Z_{\langle M\rangle_t}^\lambda$ is a solution of (26).

Set $Y_t^\lambda = -\int_0^t \operatorname{sgn} X_s^\lambda \, dX_s^\lambda$ (compare with (20)). Then,

$$\big(\sup_{s\leq t} Y_s^\lambda - Y_t^\lambda, \sup_{s\leq t} Y_s^\lambda;\, t\geq 0\big) = \big(|X_t^\lambda|, L_t(X^\lambda);\, t\geq 0\big) \tag{28}$$

which can be proved by the same argument as (24). Besides, thanks to (26), we have

$$Y_t^\lambda = -\int_0^t \operatorname{sgn} X_s^\lambda \, dX_s^\lambda = \lambda\langle M\rangle_t - \int_0^t \operatorname{sgn} X_s^\lambda \, dM_s.$$

The following statement is true for martingales $M = B_{\langle M\rangle}$ with independent B and $\langle M\rangle$ (see [7, Theorem A, Lemma (2.5)]). *If $\varepsilon = (\varepsilon_t)_{t\geq 0}$ is a predictable process taking values ± 1, then,*

$$\Big(\langle M\rangle_t, \int_0^t \varepsilon_s \, dM_s;\, t\geq 0\Big) \overset{\text{law}}{=} \big(\langle M\rangle_t, M_t;\, t\geq 0\big).$$

Consequently, $Y^\lambda \overset{\text{law}}{=} M^\lambda$. This, together with (28), completes the proof of (27).

Corollary 5. *If $M = B_{\langle M\rangle}$ is a conditionally Gaussian martingale with independent B and $\langle M\rangle$, then*

$$\big(\sup M - M, \sup M\big) \overset{\text{law}}{=} \big(|M|, L(M)\big).$$

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