INVARIANT DISTRIBUTIONS FOR SINGULAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We present necessary and sufficient conditions for the existence of a unique invariant distribution for a one-dimensional stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

such that this distribution is supported by an interval $I \subseteq \mathbb{R}$.

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1 Introduction

The problem of the existence and the uniqueness of an invariant distribution for a stochastic differential equation (abbreviated below as SDE) is a classical problem in the stochastic analysis. However, the known results (see, for example, [4]) correspond to the case, where the coefficients of the SDE "behave well enough". In particular, this means that a solution started at any point can reach any other point (we consider here only the one-dimensional case), and therefore, the invariant distribution is supported by the whole real line.

However, one often needs to consider SDEs that possess an invariant distribution supported by a subinterval of $\mathbb{R}$. For instance, this is needed in the stochastic volatility models, where the invariant distribution for a SDE describing the volatility should be carried by $\mathbb{R}_+$.

In this paper, we present necessary and sufficient conditions for the existence of a unique invariant distribution μ for the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \tag{1.1}$$

such that μ is carried by an interval $I \subseteq \mathbb{R}$ and $\operatorname{supp} \mu = \overline{I}$ (see Theorem 3.1). Here $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are measurable functions. The only assumption we impose is that $\sigma \neq 0$ at each point.

Note that if I is smaller than $\mathbb{R}$ and SDE (1.1) possesses an invariant distribution μ with $\operatorname{supp} \mu = \overline{I}$, then this SDE is *singular* in the following sense: a solution started at a point $x_0 \in I$ stays in I forever. For more information on singular SDEs, see [1]. The proof of our main theorem is largely based on results from [1].

2 Auxiliary Definitions and Known Facts

Throughout this section, we consider SDE (1.1) with a fixed starting point $x_0 \in \mathbb{R}$.

By X we denote the coordinate process on $C(\mathbb{R}_+)$, i.e. $X_t(\omega) = \omega(t)$ for $\omega \in C(\mathbb{R}_+)$; by $(\mathcal{F}_t)$ we denote the canonical filtration, i.e. $\mathcal{F}_t = \sigma(X_s;\, s \le t)$; finally, $\mathcal{F} = \sigma(X_s;\, s \ge 0)$. It will be convenient for us to define the solution of a SDE as a solution of the corresponding martingale problem. This is equivalent to the notion of a weak solution.

Definition 2.1. A *solution* of SDE (1.1) with the starting point x_0 is a measure P on $\mathcal{F}$ such that

(a) $\mathsf{P}(X_0 = x_0) = 1$;

(b) for any $t \ge 0$,

$$\int_0^t \big(|b(X_s)| + \sigma^2(X_s)\big)ds < \infty \quad \mathsf{P}\text{-a.s.};$$

(c) the process

$$M_t = X_t - \int_0^t b(X_s)ds, \quad t \ge 0 \tag{2.1}$$

is an $(\mathcal{F}_t, \mathsf{P})$-local martingale with

$$\langle M \rangle_t = \int_0^t \sigma^2(X_s)ds, \quad t \ge 0.$$

For technical reasons, we will also need the definition of a solution up to a random time.

Definition 2.2. Let S be an $(\mathcal{F}_t)$-stopping time. A *solution of* (1.1) *up to* S with the starting point x_0 is a measure P on $\mathcal{F}_S$ such that

(a) $\mathsf{P}(X_0 = x_0) = 1$;

(b) for any $t \ge 0$,

$$\int_0^{t\wedge S} \big(|b(X_s)| + \sigma^2(X_s)\big)ds < \infty \quad \mathsf{P}\text{-a.s.};$$

(c) the process

$$M_t = X_{t\wedge S} - \int_0^{t\wedge S} b(X_s)ds, \quad t \ge 0$$

is an $(\mathcal{F}_t, \mathsf{P})$-local martingale with

$$\langle M \rangle_t = \int_0^{t\wedge S} \sigma^2(X_s)ds, \quad t \ge 0.$$

(Note that in order to verify (a)–(c), it suffices to know the values of P on $\mathcal{F}_S$ only.)

Recall that a function $f:\mathbb{R}\to\mathbb{R}$ is *locally integrable at a point* a if $\int_{a-\varepsilon}^{a+\varepsilon} |f(x)|dx < \infty$ for some $\varepsilon > 0$; f is *locally integrable on a set* A if it is locally integrable at each point of A. This will be denoted as $f \in L^1_{\text{loc}}(a)$ and $f \in L^1_{\text{loc}}(A)$, respectively.

In the statements below, we use the notations

$$T_a = \inf\{t \ge 0 : X_t = a\}, \quad T_{a,c} = T_a \wedge T_c. \tag{2.2}$$

Proposition 2.3. *Suppose that* $\sigma \neq 0$ *at each point and there exists* $a \in \mathbb{R}$ *such that*

$$\frac{1+|b|}{\sigma^2} \in L^1_{\text{loc}}([a,\infty)).$$

Set

$$\rho(x) = \exp\biggl\{-\int_a^x \frac{2b(y)}{\sigma^2(y)}\,dy\biggr\}, \quad x \in [a,\infty).$$

(i) *If*

$$\int_a^\infty \rho(x)dx = \infty,$$

then, for $x_0 \in [a,\infty)$, *there exists a solution* P *up to* T_a, *and such a solution is unique. We have* $T_a < \infty$ P*-a.s.*

(ii) *If*

$$\int_a^\infty \rho(x)dx < \infty,$$

then, for $x_0 \in [a,\infty)$ *and for any* $c \geq x_0$, *there exists a solution* P^c *up to* $T_{a,c}$ *and such a solution is unique. We have* $\lim_{c\to\infty} \mathsf{P}^c(X_{T_{a,c}} = c) > 0$.

For the proof, see [1; Sect. 4.3].

A solution P up to S is called *positive* (resp., *negative*) if $\mathsf{P}(\forall t \leq S,\, X_t \geq 0) = 1$ (resp., $\mathsf{P}(\forall t \leq S,\, X_t \leq 0) = 1$).

The next proposition actually provides the one-sided classification of isolated singular points of SDEs.

Proposition 2.4. *Suppose that* $\sigma \neq 0$ *at each point and there exists* $a > 0$ *such that*

$$\frac{1+|b|}{\sigma^2} \in L^1_{\text{loc}}((0,a]).$$

Set

$$\rho(x) = \exp\biggl\{\int_x^a \frac{2b(y)}{\sigma^2(y)}\,dy\biggr\}, \quad x \in (0,a],$$

$$s(x) = \begin{cases} \displaystyle\int_0^x \rho(y)dy & \text{if } \displaystyle\int_0^a \rho(y)dy < \infty, \\ \displaystyle -\int_x^a \rho(y)dy & \text{if } \displaystyle\int_0^a \rho(x)dx = \infty, \end{cases} \qquad x \in (0,a].$$

(i) *If*

$$\int_0^a \rho(x)dx < \infty, \quad \int_0^a \frac{1+|b(x)|}{\rho(x)\sigma^2(x)}\,dx < \infty, \quad \int_0^a \frac{|b(x)|}{\sigma^2(x)}\,dx < \infty, \tag{2.3}$$

then, for $x_0 \in [0,a]$, *there exists a solution* P *up to* $T_{0,a}$, *and such a solution is unique. We have* $T_{0,a} < \infty$ P*-a.s. and* $\mathsf{P}(X_{T_{0,a}} = 0) > 0$.

(ii) *If*

$$\int_0^a \rho(x)dx < \infty, \quad \int_0^a \frac{1+|b(x)|}{\rho(x)\sigma^2(x)}\,dx < \infty, \quad \int_0^a \frac{|b(x)|}{\sigma^2(x)}\,dx = \infty, \tag{2.4}$$

then, for $x_0 \in [0, a]$, *there exists a positive solution* P *up to* T_a, *and such a solution is unique. We have* $T_a < \infty$ P*-a.s. and* $\mathsf{P}(\exists t \le T_a : X_t = 0) > 0$.

(iii) *If*

$$\int_0^a \rho(x)dx < \infty, \quad \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}\,dx = \infty, \quad \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}\,s(x)dx < \infty, \tag{2.5}$$

then, for $x_0 \in [0, a]$, *there exists a solution* P *up to* $T_{0,a}$, *and such a solution is unique. We have* $T_{0,a} < \infty$ P*-a.s. and* $\mathsf{P}(X_{T_{0,a}} = 0) > 0$. *For* $x_0 \le 0$, *any solution (up to any stopping time) is negative.*

(iv) *If*

$$\int_0^a \rho(x)dx < \infty, \; \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}dx = \infty, \; \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}s(x)dx = \infty, \; \int_0^a \frac{s(x)}{\rho(x)\sigma^2(x)}dx < \infty, \tag{2.6}$$

then, for $x_0 \in (0, a]$ *and for any* $c \in (0, x_0]$, *there exists a solution* P^c *up to* $T_{a,c}$, *and such a solution is unique. We have* $\lim_{c \downarrow 0} \mathsf{P}^c(X_{T_{a,c}} = c) > 0$. *For* $x_0 \le 0$, *any solution is negative.*

(v) *If*

$$\int_0^a \rho(x)dx < \infty, \; \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}dx = \infty, \; \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}s(x)dx = \infty, \; \int_0^a \frac{s(x)}{\rho(x)\sigma^2(x)}dx = \infty, \tag{2.7}$$

then, for $x_0 \in (0, a]$, *there exists a solution* P *up to* T_a. *Such a solution is unique and is strictly positive. We have* $\mathsf{P}(T_a = \infty$ *and* $\lim_{t \to \infty} X_t = 0) > 0$. *For* $x_0 \le 0$, *any solution is negative.*

(vi) *If*

$$\int_0^a \rho(x)dx = \infty, \quad \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}\,|s(x)|dx < \infty, \tag{2.8}$$

then, for $x_0 \in (0, a]$, *there exists a solution* P *up to* T_a. *Such a solution is unique and is strictly positive. We have* $T_a < \infty$ P*-a.s. For* $x_0 = 0$, *there exists a positive solution* P *up to* T_a, *and such a solution is unique. We have* $T_a < \infty$ P*-a.s.*

(vii) *If*

$$\int_0^a \rho(x)dx = \infty, \quad \int_0^a \frac{1 + |b(x)|}{\rho(x)\sigma^2(x)}\,|s(x)|dx = \infty, \tag{2.9}$$

then, for $x_0 \in (0, a]$, *there exists a solution* P *up to* T_a. *Such a solution is unique and is strictly positive. We have* $T_a < \infty$ P*-a.s. For* $x_0 \le 0$, *any solution is negative.*

For the proof, see [1; Sect. 2.5].

3 The Main Theorem

Let $I \subseteq \mathbb{R}$ be an interval that may be closed, open, or semi-open. We assume that I consists of more than one point. We will use the notation

$$\rho(x) = \exp\Big\{-\int^x \frac{2b(y)}{\sigma^2(y)}\,dy\Big\}, \quad s(x) = \int^x \rho(y)dy, \quad x \in \mathring{I}, \tag{3.1}$$

where $\overset{\circ}{I}$ denotes the interior of I and $\int^x$ denotes a version of the indefinite integral. This notation makes sense if $\frac{1+|b|}{\sigma^2} \in L^1_{\text{loc}}(\overset{\circ}{I})$.

For $f : I \to \mathbb{R}_+$, the statement "$\int f(x)dx < \infty$ at the endpoint of I" means that $\int_{U \cap I} f(x)dx < \infty$ for some neighborhood of this endpoint; the statement "$\int f(x)dx = \infty$ at the endpoint of I" means that $\int_{U \cap I} f(x)dx = \infty$ for any neighborhood of this endpoint (the endpoint might be finite or infinite).

Theorem 3.1. *Suppose that $\sigma \neq 0$ at each point. Then* (i) + (ii) $\Leftrightarrow$ (a) + $\cdots$ + (e)*:*

(i) *For any starting point $x_0 \in I$, there exists a solution P_{x_0} of* (1.1) *with $\mathsf{P}_{x_0}(\forall t \geq 0,\ X_t \in I) = 1$, and such a solution is unique.*

(ii) *There exists an invariant distribution μ (i.e. for any $t \geq 0$, $\text{Law}(X_t \mid \mathsf{P}_\mu) = \mu$, where $\mathsf{P}_\mu = \int_I \mathsf{P}_x \mu(dx)$) carried by I (i.e. $\mu(I) = 1$) with $\text{supp}\,\mu = \overline{I}$ ($\overline{I}$ denotes the closure of I), and such a distribution is unique.*

(a) *We have*

$$\frac{1+|b|}{\sigma^2} \in L^1_{\text{loc}}(\overset{\circ}{I}). \tag{3.2}$$

(b) *We have*

$$\frac{1}{\rho\sigma^2} \in L^1(I) \tag{3.3}$$

(i.e. $\int_I \rho^{-1}(x)\sigma^{-2}(x)dx < \infty$).

(c) *At the infinite endpoints of I, we have*

$$\int \rho(x)dx = \infty. \tag{3.4}$$

(d) *At the finite endpoints of I that do not belong to I, we have*

$$\int \rho(x)dx = \infty. \tag{3.5}$$

(e) *At the finite endpoints of I that belong to I, we have either*

$$\int \rho(x)dx < \infty, \quad \int \frac{1+|b(x)|}{\rho(x)\sigma^2(x)}dx < \infty, \quad \int \frac{|b(x)|}{\sigma^2(x)}dx = \infty \tag{3.6}$$

or

$$\int \rho(x)dx = \infty, \quad \int \frac{1+|b(x)|}{\rho(x)\sigma^2(x)}|s(x)|dx < \infty. \tag{3.7}$$

If these conditions are satisfied, then the measure μ given by (ii) *has the form*

$$\mu(dx) = \frac{c}{\rho(x)\sigma^2(x)}dx, \tag{3.8}$$

where c is the normalizing constant. Moreover, for any distribution ν carried by I, we have

$$\text{Law}(X_t \mid \mathsf{P}_\nu) \xrightarrow[t\to\infty]{\text{Var}} \mu,$$

where $\mathsf{P}_\nu = \int_I \mathsf{P}_x \nu(dx)$.

Remark. In the case, where $I = \mathbb{R}$, condition (3.2) is the Engelbert–Schmidt condition that guarantees the existence of a unique (possibly, exploding) solution of (1.1) (see [3]). □

Corollary 3.2 (Cox–Ingersoll–Ross process). *Let $b, c, \sigma > 0$. Consider the SDE*

$$dX_t = (b - cX_t)dt + \sigma\sqrt{X_t}dB_t. \tag{3.9}$$

For any starting point $x_0 \geq 0$, there exists a positive solution of (3.9), and such a solution is unique. There exists an invariant distribution supported by $\mathbb{R}_+$, and such a distribution is unique. It is given by

$$\mu(dx) = c\, x^{2b/\sigma^2-1}\, e^{-2bx/\sigma^2}\, I(x > 0)\, dx,$$

where c is the normalizing constant. For any probability measure ν carried by $\mathbb{R}_+$, we have

$$\mathrm{Law}(X_t \mid \mathsf{P}_\nu) \xrightarrow[t\to\infty]{\mathrm{Var}} \mu.$$

Proof of Theorem 3.1. **(a)** + ⋯ + **(e)** ⇒ **(i)** Let us first prove the existence of P_{x_0}. We will do this only for the case, where I is a compact interval, condition (3.6) is satisfied at the left endpoint of I, and condition (3.7) is satisfied at the right endpoint of I. For the other cases, the proofs are similar.

Suppose first that x_0 does not coincide with the right endpoint of I. Without loss of generality, we can assume that the left endpoint of I is 0 and that $s(0) = 0$, where s is given by (3.1). Let B be a Brownian motion started at $s(x_0)$. Set

$$\varphi_t = \inf\Big\{u \geq 0 : \int_0^u I(B_s > 0)ds > t\Big\}, \quad U_t = B_{\tau_t}, \quad t \geq 0.$$

The process U is known to be a Brownian motion reflected at zero (see [5; § 2.11]), and hence,

$$U_t = s(x_0) + W_t + L_t^0(U), \quad t \geq 0,$$

where W is an $(\mathcal{F}_t^U)$-Brownian motion and $L_t^0(U)$ denotes the local time spent by U at 0 by the time t (see [6; Ch. VI, Th. 1.2]).

Set

$$\varkappa(y) = \rho(s^{-1}(y))\sigma(s^{-1}(y)), \quad y \in s(I) \tag{3.10}$$

and consider

$$\psi_t = \inf\Big\{u \geq 0 : \int_0^u \frac{1}{\varkappa^2(U_s)}ds > t\Big\}, \quad t \geq 0,$$
$$V_t = U_{\psi_t} = s(x_0) + W_{\psi_t} + L_{\psi_t}^0(U), \quad t \geq 0.$$

The process $M_t = W_{\psi_t}$ is a continuous $(\mathcal{F}_{\tau_t+}^U)$-local martingale (here $\mathcal{F}_{t+}^U = \bigcap_{\varepsilon>0}\sigma(U_s;\ s \leq t+\varepsilon))$ with

$$\langle M\rangle_t = \psi_t = \int_0^{\psi_t} \frac{\varkappa^2(U_s)}{\varkappa^2(U_s)}\, ds = \int_0^t \varkappa^2(V_s)ds, \quad t \geq 0$$

(see [6; Ch. V, Prop. 1.5]).

Let us consider the function

$$f(y) = \begin{cases} s^{-1}(y) & \text{if } y > 0, \\ 0 & \text{if } y \le 0. \end{cases}$$

The functions f, f' are absolutely continuous on $(0,\infty)$ and

$$f'(y) = \frac{1}{\rho(s^{-1}(y))}, \quad f''(y) = \frac{2b(s^{-1}(y))}{\varkappa^2(y)}, \quad y > 0.$$

Furthermore, for any $a > 0$,

$$\int_0^a \frac{|2b(s^{-1}(y))|}{\varkappa^2(y)}\,dy = \int_0^{s^{-1}(a)} \frac{|2b(x)|}{\rho(x)\sigma^2(x)}\,dx < \infty. \tag{3.11}$$

Hence, f' has bounded variation on compact intervals. Moreover, (3.11) shows that there exists a limit $\lim_{y\downarrow 0} f'(y) = \lim_{x\downarrow 0}\frac{1}{\rho(x)}$. Taking into account (3.6), we deduce that this limit equals zero. By the Itô-Tanaka formula and the occupation times formula,

$$\begin{aligned} f(V_t) &= \int_0^t f'_-(V_s)dM_s + \int_0^t f'_-(V_s)dL^0_{\psi_s}(U) + \frac{1}{2}\int_0^\infty \frac{2b(s^{-1}(y))}{\varkappa^2(y)}\,L^y_t(V)dy \\ &= N_t + \int_0^{\psi_t} f'_-(U_s)dL^0_s(U) + \int_0^t b(s^{-1}(V_s))ds = N_t + \int_0^t b(f(V_s))ds, \quad t \ge 0 \end{aligned}$$

(in the last step we used the equality $f'_-(0) = 0$). Here N is a continuous $(\mathcal{F}^V_t)$-local martingale with

$$\langle N\rangle_t = \int_0^t \frac{\varkappa^2(V_s)}{\rho^2(s^{-1}(V_s))}\,ds = \int_0^t \sigma^2(f(V_s))ds, \quad t \ge 0.$$

As a result, the measure $\mathsf{P} = \mathrm{Law}(f(V_t);\, t \ge 0)$ is a solution of (1.1).

Suppose now that x_0 coincides with the right endpoint r of I. Fix $a \in \mathring{I}$. By Proposition 2.4 (vi), there exists a solution Q up to T_a with the starting point r (T_a is given by (2.2)). It follows from the reasoning above that there exists a solution R with the starting point a. Let P_r be the image of $\mathsf{Q}\times\mathsf{R}$ under the map

$$C(\mathbb{R}_+)\times C(\mathbb{R}_+) \ni (\omega_1,\omega_2) \longmapsto G(\omega_1,\omega_2,T_a(\omega_1)) \in C(\mathbb{R}_+),$$

where G is the gluing function defined by

$$G(\omega_1,\omega_2,T)(t) = \begin{cases} \omega_1(t) & \text{if } t < T, \\ \omega_2(t-T) & \text{if } t \ge T. \end{cases}$$

Then P_r is a solution with the starting point r (for more details, see [1; Lemma B.10]).

Let us now prove the uniqueness of P_{x_0}. Without loss of generality, we assume that x_0 is not the right endpoint of I (otherwise, it is not the left endpoint). Suppose that there exist two different solutions P_{x_0} and $\widetilde{\mathsf{P}}_{x_0}$. Fix $a \in \mathring{I}$ such that $a > x_0$. It follows from Propositions 2.3, 2.4 that $\mathsf{P}_{x_0} \mid \mathcal{F}_{T_a} = \widetilde{\mathsf{P}}_{x_0} \mid \mathcal{F}_{T_a}$ and $T_a < \infty$ P_{x_0}, $\widetilde{\mathsf{P}}_{x_0}$-a.s. Let $(\mathsf{Q}_\omega)_{\omega\in C(\mathbb{R}_+)}$ (resp., $(\widetilde{\mathsf{Q}}_\omega)_{\omega\in C(\mathbb{R}_+)}$) denote the conditional P_{x_0}-distribution (resp., $\widetilde{\mathsf{P}}_{x_0}$-distribution) given $\mathcal{F}_{T_a}$. Let R_ω (resp., $\widetilde{\mathsf{R}}_\omega$) be the image of Q_ω (resp., $\widetilde{\mathsf{Q}}_\omega$) under the map

$$C(\mathbb{R}_+) \ni f \longmapsto g \in C(\mathbb{R}_+), \quad g(t) = f(t + T_a(f)).$$

Then, for P_{x_0}-a.e. (resp., $\widetilde{\mathsf{P}}_{x_0}$-a.e.) ω, the measure R_ω (resp., $\widetilde{\mathsf{R}}_\omega$) is a solution of (1.1) with the starting point a. It follows from Propositions 2.3, 2.4 that there exists a unique solution R up to T_{x_0} with the starting point a. Hence, for P_{x_0}-a.e. (resp., $\widetilde{\mathsf{P}}_{x_0}$-a.e.) ω, we have $\mathsf{R}_\omega \mid \mathcal{F}_{T_{x_0}} = \mathsf{R}$ (resp., $\widetilde{\mathsf{R}}_\omega \mid \mathcal{F}_{T_{x_0}} = \mathsf{R}$). As a result, $\mathsf{P}_{x_0} \mid \mathcal{F}_S = \widetilde{\mathsf{P}}_{x_0} \mid \mathcal{F}_S$, where $S = \inf\{t \ge T_a : X_t = x_0\}$. Proceeding in the same way, we conclude that, for any n, $\mathsf{P}_{x_0} \mid \mathcal{F}_{S_n} = \widetilde{\mathsf{P}}_{x_0} \mid \mathcal{F}_{S_n}$, where $S_0 = S$,

$$S_{n+1} = \inf\{t \ge \tau_{n+1} : X_t = x_0\}, \quad \tau_{n+1} = \inf\{t \ge S_n : X_t = a\}.$$

Since $S_n \xrightarrow[n\to\infty]{} \infty$, we conclude that $\mathsf{P}_{x_0} = \widetilde{\mathsf{P}}_{x_0}$.

(a) + ⋯ + (e) ⇒ (ii) Suppose first that $x_0 \in \overset{\circ}{I}$ and let P_{x_0} be the solution given by (i). It follows from the explicit construction of the solution given above that

$$\mathsf{P}_{x_0} = \mathrm{Law}\big(s^{-1}(B_{\tau_t});\, t \ge 0\big), \tag{3.12}$$

where B is a Brownian motion started at $s(x_0)$, s is given by (3.1),

$$\tau_t = \left\{u \ge 0 : \int_0^u \frac{I(B_s \in J)}{\varkappa^2(B_s)}\, ds > t\right\}, \quad t \ge 0,$$

$J = s(I)$, and $\varkappa$ is given by (3.10). Consider the measure ν on J defined as $\nu(dy) = \frac{c}{\varkappa^2(y)}\, dy$, where c is the normalizing constant. Such a constant exists in view of (3.3) and the equality

$$\int_J \frac{1}{\varkappa^2(y)}\, dy = \int_I \frac{1}{\rho(x)\sigma^2(x)}\, dx.$$

It is known (see [7; Th. 54.5]) that $\mathrm{Law}(B_{\tau_t}) \xrightarrow[t\to\infty]{\mathrm{Var}} \nu$. Hence,

$$\mathrm{Law}(X_t \mid \mathsf{P}_{x_0}) \xrightarrow[t\to\infty]{\mathrm{Var}} \mu, \tag{3.13}$$

where μ is the measure on I given by (3.8).

Suppose now that x_0 is an endpoint of I. For any $a \in I$ and $t \ge 0$, we have

$$\int_0^t I(X_s = a) ds = \int_0^t \frac{I(X_s = a)}{\sigma^2(X_s)}\, d\langle X\rangle_s = \int_I \frac{I(x = a)}{\sigma^2(x)}\, L_t^x(X) dx = 0 \quad \mathsf{P}_{x_0}\text{-a.s.} \tag{3.14}$$

Hence, for a.e. $t \ge 0$, $\mathsf{P}_{x_0}(X_t = a) = 0$. Combining this with the Markov property of $(\mathsf{P}_x)_{x\in I}$ (see [8; Th. 6.2]), we deduce that (3.13) holds for any $x_0 \in I$. For any $s \ge 0$, $t \ge 0$, we have, by the Markov property,

$$\mathrm{Law}(X_{t+s} \mid \mathsf{P}_\mu) = \mathrm{Law}(X_s \mid \mathrm{Law}(X_t \mid \mathsf{P}_\mu)).$$

Letting $t \to \infty$ and keeping (3.13) in mind, we get $\mathrm{Law}(X_s \mid \mathsf{P}_\mu) = \mu$, so that μ is invariant. The uniqueness of an invariant distribution is an immediate consequence of (3.13).

(i) + (ii) ⇒ (a) Let us suppose that there exists $a \in \overset{\circ}{I}$ such that

$$\frac{1+|b|}{\sigma^2} \notin L^1_{\mathrm{loc}}(a). \tag{3.15}$$

Without loss of generality, we can assume that $a = 0$.

Let P_0 be the solution given by (i). First we will verify that for any $t \ge 0$, $L^0_t(X) = 0$ P-a.s. (compare with [2; Th. 2.3]). We have

$$\int_0^t I(X_s = 0)dX_s = \int_0^t I(X_s = 0)b(X_s)ds + \int_0^t I(X_s = 0)dM_s,$$

where M is defined in (2.1). The process $N_t = \int_0^t I(X_s = 0)dM_s$ is a continuous $(\mathcal{F}_t, \mathsf{P}_0)$-local martingale with

$$\langle N\rangle_t = \int_0^t I(X_s = 0)\sigma^2(X_s)ds.$$

Recalling (3.14), we deduce that $\int_0^t I(X_s = 0)dX_s = 0$ P_0-a.s., and hence (see [6; Ch. VI, Th. 1.7]), $L^0_t(X) = L^{0-}_t(X)$ P_0-a.s. Combining this with the equality

$$\int_0^t (1 + |b(X_s)|)ds = \int_0^t \frac{1+|b(X_s)|}{\sigma^2(X_s)}\,d\langle X\rangle_s = \int_{\mathbb{R}} \frac{1+|b(x)|}{\sigma^2(x)}\,L^x_t(X)dx \quad \mathsf{P}_0\text{-a.s.},$$

we get $L^0_t(X) = 0$ P_0-a.s.

Suppose now that

$$\mathsf{P}_0(\exists t > 0 : X_t > 0) > 0, \quad \mathsf{P}_0(\exists t > 0 : X_t < 0) > 0. \tag{3.16}$$

By the Tanaka formula,

$$X^+_t = \int_0^t I(X_s > 0)b(X_s)ds + \int_0^t I(X_s > 0)dM_s, \quad t \ge 0.$$

Set

$$A_t = \int_0^t I(X_s > 0)ds, \quad \tau_t = \inf\{s \ge 0 : A_s > t\}, \quad t \ge 0$$

and consider

$$Y_t = \begin{cases} X^+_{\tau_t} & \text{if } t < A_\infty,\\ 0 & \text{if } t \ge A_\infty.\end{cases}$$

We have

$$Y_t = \int_0^{\tau_t} I(X_s > 0)b(X_s)ds + \int_0^{\tau_t} I(X_s > 0)dM_s = \int_0^{t\wedge A_\infty} b(Y_s)ds + K_t, \quad t \ge 0.$$

The process K is a continuous $(\mathcal{F}_{\tau_t}, \mathsf{P}_0)$-local martingale with

$$\langle K\rangle_t = \int_0^{\tau_t} I(X_s > 0)\sigma^2(X_s)ds = \int_0^{t\wedge A_\infty} \sigma^2(Y_s)ds, \quad t \ge 0.$$

Now, let (Y^1, A^1_∞), $(Y^2, A^2_\infty), \dots$ be independent copies of (Y, A_∞). Set

$$Z_t = \begin{cases} Y^1_t & \text{if } t < A^1_\infty, \\ Y^2_{t-A^1_\infty} & \text{if } A^1_\infty \le t < A^1_\infty + A^2_\infty, \\ \dots \end{cases}$$

One can verify that the measure $\widetilde{\mathsf{P}}_0 = \mathrm{Law}(Z_t;\, t \ge 0)$ is a positive solution of (1.1) with the starting point 0. It follows from (i) that $\widetilde{\mathsf{P}}_0 = \mathsf{P}_0$, but this contradicts (3.16). As a result, P_0 should be either positive or negative.

Without loss of generality, we can assume that P_0 is positive. Let μ be the invariant distribution given by (ii). Set

$$\overline{\mu}(dx) = I(x \ge 0)\mu(dx), \quad \overline{\mu}_t = \int_I \mathrm{Law}(X_t \mid \mathsf{P}_x)\overline{\mu}(dx).$$

Since P_0 is positive, we deduce using the strong Markov property of $(\mathsf{P}_x)_{x\in I}$ (see [8; Th. 6.2]) that the mass $\overline{\mu}$ cannot escape from $I \cap \mathbb{R}_+$. Consequently, $\overline{\mu}_t(\mathbb{R}_+) \ge \overline{\mu}(\mathbb{R}_+)$. On the other hand, since μ is invariant, $\overline{\mu}_t|_{\mathbb{R}_+} \le \mu|_{\mathbb{R}_+} = \overline{\mu}|_{\mathbb{R}_+}$. Hence, $\overline{\mu}_t = \overline{\mu}$. This means that any distribution of the form $\alpha\overline{\mu} + \beta\mu$ is invariant. The contradiction shows that (3.15) is false.

(i)+(ii) $\Rightarrow$ (c) Suppose that the right endpoint of I is $+\infty$ and $\int \rho(x)dx < \infty$ in the neighborhood of $+\infty$. Take any $x_0 \in \overset{\circ}{I}$. It follows from Proposition 2.3 (ii) that $\mathsf{P}_{x_0}(\lim_{t\to\infty} X_t = \infty) > 0$. For any $y \ge x_0$, we have by the strong Markov property of $(\mathsf{P}_x)_{x\in I}$:

$$\mathsf{P}_{x_0}\Big(\lim_{t\to\infty} X_t = \infty\Big) = \mathsf{P}_{x_0}(T_y < \infty)\mathsf{P}_y\Big(\lim_{t\to\infty} X_t = \infty\Big)$$

(T_y is given by (2.2)). Hence, for any $y \ge x_0$,

$$\mathsf{P}_y\Big(\lim_{t\to\infty} X_t = \infty\Big) \ge \mathsf{P}_{x_0}\Big(\lim_{t\to\infty} X_t = \infty\Big).$$

But this contradicts the existence of an invariant distribution supported by $\overline{I}$.

(i)+(ii) $\Rightarrow$ (d) Suppose that the left endpoint of I equals 0 and does not belong to I. It is clear from Proposition 2.4 that none of conditions (2.3)–(2.6) is satisfied since otherwise the condition $\mathsf{P}_{x_0}(\forall t \ge 0,\, X_t > 0) = 1$ will be violated. If condition (2.7) is satisfied, then $\mathsf{P}_{x_0}(\lim_{t\to\infty} X_t = 0) > 0$, and, applying the same reasoning as above, we arrive at a contradiction with (ii).

(i) + (ii) $\Rightarrow$ (e) Suppose that the left endpoint of I equals 0 and belongs to I. It is clear from Proposition 2.4 that none of conditions (2.5), (2.6), (2.7), and (2.9) is satisfied, since otherwise a solution with the starting point 0 will be negative.

Suppose that condition (2.3) is satisfied. Note that in this case

$$\int \frac{1 + |b(x)|}{\sigma^2(x)}\, dx < \infty$$

at zero. Without loss of generality, we can assume that $s(0) = 0$, where s is given by (3.1). Consider the function

$$f(x) = \begin{cases} s(x) & \text{if } x > 0, x \in I, \\ \rho(0+)x & \text{if } x \le 0. \end{cases}$$

By the Itô-Tanaka formula,

$$f(X_t)=\int_0^t \rho(X_s)b(X_s)ds+\int_0^t \rho(X_s)dM_s-\frac{1}{2}\int_{\mathbb{R}}\frac{2b(x)}{\sigma^2(x)}\rho(x)L_t^x(X)dx=\int_0^t \rho(X_s)dM_s.$$

Thus, $f(X)$ is an $(\mathcal{F}_t,\mathsf{P}_0)$-local martingale, where P_0 is given by (i). Since $f(X_0)=0$ and $f(X)\ge 0$, we get $f(X)=0$ P_0-a.s., which means that $X=0$ P_0-a.s., but this contradicts (3.14). As a result, condition (2.3) is not satisfied.

(i) + (ii) $\Rightarrow$ (b) We have proved that (i) + (ii) $\Rightarrow$ (a) + (c) + (d) + (e). The reasoning in the proof of the implication (a) $+\dots+$ (e) $\Rightarrow$ (i) (where we actually used only (a), (c), (d), and (e)) shows that, for $x_0\in \overset{\circ}{I}$, (3.12) is satisfied.

Choose functions $f:J\to\mathbb{R}_+$, $g:J\to\mathbb{R}_+$ such that f and g are bounded, $f\le \alpha g$ for some $\alpha\in\mathbb{R}$, and

$$\int_J \frac{f(s^{-1}(y))}{\varkappa^2(y)}\,dy<\infty,\qquad 0<\int_J \frac{g(s^{-1}(y))}{\varkappa^2(y)}\,dy<\infty.$$

It follows from the ergodic theorem for one-dimensional diffusions (see [7; Ch. V, Th. 53.1]) that

$$\frac{\int_0^t f(s^{-1}(B_{\tau_s}))ds}{\int_0^t g(s^{-1}(B_{\tau_s}))ds}\xrightarrow[t\to\infty]{\text{a.s.}}\frac{\int_J \frac{f(s^{-1}(y))}{\varkappa^2(y)}\,dy}{\int_J \frac{g(s^{-1}(y))}{\varkappa^2(y)}\,dy}=\frac{\int_I \frac{f(x)}{\rho(x)\sigma^2(x)}\,dx}{\int_I \frac{g(x)}{\rho(x)\sigma^2(x)}\,dx}.$$

Hence,

$$\frac{\int_0^t f(X_s)ds}{\int_0^t g(X_s)ds}\xrightarrow[t\to\infty]{\mathsf{P}_\mu\text{-a.s.}}\frac{\int_I \frac{f(x)}{\rho(x)\sigma^2(x)}\,dx}{\int_I \frac{g(x)}{\rho(x)\sigma^2(x)}\,dx}. \tag{3.17}$$

On the other hand,

$$\mathsf{E}_{\mathsf{P}_\mu}\frac{1}{t}\int_0^t f(X_s)ds=\int_I f(x)\mu(dx),$$
$$\mathsf{E}_{\mathsf{P}_\mu}\frac{1}{t}\int_0^t g(X_s)ds=\int_I g(x)\mu(dx).$$

This, combined with (3.17) and with the properties of f and g, shows that

$$\int_I f(x)\mu(dx)=\frac{\int_I \frac{f(x)}{\rho(x)\sigma^2(x)}\,dx}{\int_I \frac{g(x)}{\rho(x)\sigma^2(x)}\,dx}\int_I g(x)\mu(dx).$$

It follows that

$$\mu(dx)=\frac{c}{\rho(x)\sigma^2(x)}\,dx$$

with some constant c, which implies (b). □

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