LIMIT BEHAVIOUR OF THE "HORIZONTAL-VERTICAL" RANDOM WALK AND SOME EXTENSIONS OF THE DONSKER-PROKHOROV INVARIANCE PRINCIPLE

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Abstract. We consider a two-dimensional random walk that moves in the horizontal direction on the half-plane $\{y > x\}$ and in the vertical direction on the half-plane $\{y \le x\}$. The limit behaviour (as the time interval between two steps and the size of each step tend to zero) of this "horizontal-vertical" random walk is investigated.

In order to solve this problem, we prove an extension of the Donsker-Prokhorov invariance principle. The extension states that the discrete-time stochastic integrals with respect to the appropriately renormalized one-dimensional random walk converge in distribution to the corresponding stochastic integral with respect to a Brownian motion.

This extension enables us to construct a discrete-time approximation of the local time of a Brownian motion.

We also provide discrete-time approximations of skew Brownian motions.

Key words and phrases. Limit theorems for degenerate processes, Donsker-Prokhorov invariance principle, local time of Brownian motion, skew Brownian motions, Skorokhod embedding problem.

1 Introduction

1. Limit behaviour of the "horizontal-vertical" random walk. Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathsf{E}\xi_k = 0$, $\mathsf{E}\xi_k^2 = 1$. We construct a two-dimensional "horizontal-vertical" random walk $(Z_k;\, k \in \mathbb{Z}_+) = (X_k, Y_k;\, k \in \mathbb{Z}_+)$ by the following procedure: $X_0 = 0$, $Y_0 = 0$,

$$X_{k+1} = \begin{cases} X_k + \xi_{k+1} & \text{if } Y_k > X_k, \\ X_k & \text{if } Y_k \le X_k, \end{cases}$$

$$Y_{k+1} = \begin{cases} Y_k & \text{if } Y_k > X_k, \\ Y_k - \xi_{k+1} & \text{if } Y_k \le X_k. \end{cases}$$

In other words, Z_{k+1} is obtained from Z_k by the shift whose modulus equals $|\xi_{k+1}|$. If Z_k belongs to the half-plane $\{y > x\}$, then the shift occurs in the horizontal direction. If Z_k belongs to the half-plane $\{y \le x\}$, then the shift occurs in the vertical direction (see Figure 1). For each $n \in \mathbb{N} = \{1, 2, \dots\}$, we consider

$$Z^n_{k/n} = \frac{1}{\sqrt{n}} Z_k, \quad k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$$

and construct the process $(Z^n_t;\, t \ge 0)$ by the linear interpolation of $(Z^n_{k/n};\, k \in \mathbb{Z}_+)$.

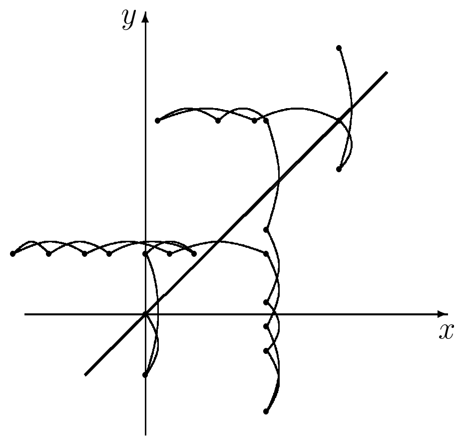


Figure 1. The "horizontal-vertical" random walk

The following question arises: What is the limit behaviour (as $n \to \infty$) of the process $(Z^n_t;\, t \ge 0)$?

We prove in Section 5 that the sequence of processes $(Z^n_t;\, t \ge 0)$ converges in distribution (as $n \to \infty$) to a process $(Z_t;\, t \ge 0)$ and give the explicit form of this process (Theorem 5.1).

We also present another equivalent construction of the limit process $(Z_t;\, t \ge 0)$. This construction shows, in particular, that the paths of Z yield an interesting and transparent representation of the Brownian excursions.

2. Extensions of the Donsker-Prokhorov invariance principle. In order to solve the above problem, we provide in Section 2 the following extension of the Donsker-Prokhorov invariance principle.

Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathsf{E}\xi_k = 0$, $\mathsf{E}\xi_k^2 = 1$. Let $f : \mathbb{R} \to \mathbb{R}^d$ be a Borel function. For each $n \in \mathbb{N}$, we set $\xi_k^n = \frac{1}{\sqrt{n}}\xi_k$ and consider

$$X_{k/n}^n = \sum_{i=1}^{k} \xi_i^n, \qquad k \in \mathbb{Z}_+,$$
$$Y_{k/n}^n = \sum_{i=1}^{k} f\big(X_{(i-1)/n}^n\big)\xi_i^n, \quad k \in \mathbb{Z}_+.$$

Construct the processes $(X_t^n;\, t \geq 0)$, $(Y_t^n;\, t \geq 0)$ by linear interpolation of $(X_{k/n}^n;\, k \in \mathbb{Z}_+)$, $(Y_{k/n}^n;\, k \in \mathbb{Z}_+)$. Then, under some regularity conditions imposed on f and ξ_k,

$$(X_t^n, Y_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{Law}} \Big(B_t, \int_0^t f(B_s)dB_s;\, t \geq 0\Big),$$

where B is a Brownian motion started at zero (Theorems 2.2, 2.3). The sign $\xrightarrow{\text{Law}}$ here corresponds to the weak convergence of probability measures on $C(\mathbb{R}_+, \mathbb{R}^{d+1})$ endowed with the topology of uniform convergence on compact intervals. Note that, with no regularity conditions on f and ξ_k, this result is not true (see Example 2.4).

The convergence of stochastic integrals in a more general situation (for a semimartingale instead of a Brownian motion and for arbitrary integrands instead of $f(B)$) is studied in the book [9; Ch. IX, §5b] by J. Jacod and A.N. Shiryaev. However, these general results cannot be applied to our situation in the case, where f is not continuous.

The method used to prove Theorems 2.2, 2.3 is based on the Skorokhod embedding problem. This method is well known in limit theorems. For instance, the book [3; Ch. 13, §5] by L. Breiman contains a simple proof of the Donsker-Prokhorov invariance principle that employs this method. B. Cadre [5] provided a discrete-time approximation of the intersection local time of a two-dimensional Brownian motion employing Skorokhod's embedding problem.

3. Approximation of the Brownian local time. Theorem 2.2 yields the following corollary (see Section 3).

Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathsf{P}(\xi_k = 1) = \mathsf{P}(\xi_k = -1) = \frac{1}{2}$. Let us set

$$X_k = \sum_{i=1}^{k} \xi_i, \qquad k \in \mathbb{Z}_+,$$
$$L_k = \sum_{i=0}^{k-1} I(X_i = 0), \quad k \in \mathbb{Z}_+.$$

For each $n \in \mathbb{N}$, we consider

$$X_{k/n}^n = \frac{1}{\sqrt{n}}X_k, \quad L_{k/n}^n = \frac{1}{\sqrt{n}}L_k, \quad k \in \mathbb{Z}_+$$

and construct the processes $(X_t^n;\, t \geq 0)$, $(L_t^n;\, t \geq 0)$ by linear interpolation of $(X_{k/n}^n;\, k \in \mathbb{Z}_+)$, $(L_{k/n}^n;\, k \in \mathbb{Z}_+)$. Then

$$(X_t^n, L_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{Law}} (B_t, L_t;\, t \geq 0),$$

where B is a Brownian motion started at zero and L is its local time at zero.

4. Approximations of skew Brownian motions. The method based on the Skorokhod embedding problem enables us to prove one more extension of the Donsker-Prokhorov invariance principle (see Section 4).

Let $(X_k;\, k \in \mathbb{Z}_+)$ be an integer-valued Markov chain with $X_0 = 0$ and the transition probabilities

$$\mathsf{P}(X_{k+1} = i+1 \mid X_k = i) = \frac{1}{2}, \quad \mathsf{P}(X_{k+1} = i-1 \mid X_k = i) = \frac{1}{2} \quad \text{if } i \neq 0,$$
$$\mathsf{P}(X_{k+1} = 1 \mid X_k = 0) = p, \qquad \mathsf{P}(X_{k+1} = -1 \mid X_k = 0) = 1-p,$$

where $p \in [0,1]$. Let $f : \mathbb{R} \to \mathbb{R}^d$ be a Borel function. For each $n \in \mathbb{N}$, we consider

$$X^n_{k/n} = \frac{1}{\sqrt{n}} X_k, \qquad\qquad k \in \mathbb{Z}_+,$$
$$Y^n_{k/n} = \sum_{i=1}^{k} f\big(X^n_{(i-1)/n}\big)\big(X^n_{i/n} - X^n_{(i-1)/n}\big), \quad k \in \mathbb{Z}_+$$

and construct the processes $(X^n_t;\, t \geq 0)$, $(Y^n_t;\, t \geq 0)$ by linear interpolation of $(X^n_{k/n};\, k \in \mathbb{Z}_+)$, $(Y^n_{k/n};\, k \in \mathbb{Z}_+)$. Then, under some regularity conditions imposed on f,

$$(X^n_t, Y^n_t;\, t \geq 0) \xrightarrow[n\to\infty]{\text{Law}} \left(B^p_t, \int_0^t f(B^p_s)dB^p_s;\, t \geq 0\right), \tag{1.1}$$

where B^p is a skew Brownian motion with parameter p started at zero.

This result provides discrete-time approximations of skew Brownian motions. It completes the result of J.M. Harrison and L.A. Shepp [6] who proved the convergence of the marginal distributions, which we denote by $X^n_t \xrightarrow[n\to\infty]{\text{Law}} B^p_t$, $t \geq 0$. The convergence of the first components in (1.1) follows from the paper [4] by J.K. Brooks and R.V. Chacon. However, we use another (simpler) method.

There also exist other approximations of skew Brownian motions. W.A. Rosenkrantz [11] provided an approximation of a skew Brownian motion by appropriately renormalized solutions of the stochastic differential equation $dX_t = b(X_t)dt + dB_t$.

2 Extensions of the Donsker-Prokhorov Invariance Principle

1. The results. Let $(\xi_k)_{k=1}^\infty$ be a sequence of i.i.d. random variables with $\mathsf{E}\xi_k = 0$, $\mathsf{E}\xi_k^2 = 1$. Let $f : \mathbb{R} \to \mathbb{R}^d$ be a Borel function. For each $n \in \mathbb{N}$, we set $\xi^n_k = \frac{1}{\sqrt{n}}\xi_k$ and consider

$$X^n_{k/n} = \sum_{i=1}^{k} \xi^n_i, \qquad\qquad k \in \mathbb{Z}_+, \tag{2.1}$$

$$Y^n_{k/n} = \sum_{i=1}^{k} f\big(X^n_{(i-1)/n}\big)\xi^n_i, \quad k \in \mathbb{Z}_+. \tag{2.2}$$

Construct the processes $(X^n_t;\, t \geq 0)$, $(Y^n_t;\, t \geq 0)$ by linear interpolation of $(X^n_{k/n};\, k \in \mathbb{Z}_+)$, $(Y^n_{k/n};\, k \in \mathbb{Z}_+)$.

Definition 2.1. A function f is *piecewise continuous* if there exists a collection of disjoint intervals $(J_k)_{k=1}^{\infty}$ (each J_k may be closed, open or semi-open; it may also consist of one point) with the following properties:

i) $\bigcup_{k=1}^{\infty} J_k = \mathbb{R}$ and, for any compact interval J, there exists $m \in \mathbb{N}$ such that $\bigcup_{k=1}^{m} J_k \supseteq J$;

ii) the restriction of f to each J_k is continuous on J_k and has finite limits at those endpoints of J_k that do not belong to J_k.

`Remark`. Any piecewise continuous function is locally bounded. □

Theorem 2.2. *If f is piecewise continuous, then*

$$(X_t^n, Y_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{Law}} \Big(B_t, \int_0^t f(B_s)dB_s;\, t \geq 0\Big), \tag{2.3}$$

where B is a Brownian motion started at zero.

Theorem 2.3. *If f is locally bounded and there exists $m \in \mathbb{N}$ such that the distribution of $\xi_1 + \dots + \xi_m$ has an absolutely continuous* (*with respect to the Lebesgue measure*) *component, then* (2.3) *holds.*

`Remark`. Theorem 2.2 implies the Donsker-Prokhorov invariance principle (for the sums of identically distributed random variables). □

The following example shows that the regularity conditions imposed on f and ξ_k in Theorems 2.2, 2.3 are essential.

Example 2.4. Let $(\xi_k)_{k=1}^{\infty}$ be i.i.d. random variables with $\mathsf{P}(\xi_k = 1) = \mathsf{P}(\xi_k = -1) = \frac{1}{2}$. Let $A = \big\{\frac{m}{\sqrt{n}};\, m \in \mathbb{Z}, n \in \mathbb{N}\big\}$ and set $f = I_{\mathbb{R}\setminus A}$. Then all the random variables $X_{k/n}^n$ take values in A, and hence, $Y_{k/n}^n = 0$. On the other hand,

$$\int_0^t f(B_s)dB_s = B_t, \quad t \geq 0,$$

and therefore, (2.3) does not hold.

2. The proofs. Theorem 2.2 follows from Lemma 2.8 given below. Theorem 2.3 follows from Lemma 2.10.

Proposition 2.5 (Skorokhod). *Let ξ be a random variable with $\mathsf{E}\xi = 0$, $\mathsf{E}\xi^2 < \infty$. Let $(B_t;\, t \geq 0)$ be a Brownian motion started at zero. Then there exists a $(\mathcal{F}_t^B)$-stopping time τ such that $\mathsf{E}\tau = \mathsf{E}\xi^2$ and $\mathrm{Law}(B_\tau) = \mathrm{Law}(\xi)$.*

Many solutions of this problem have been obtained; see, for example, [10; Ch. VI, (5.4)].

Lemma 2.6. *Let $(B_t;\, t \geq 0)$ be a Brownian motion started at zero. There exists a collection $(\tau_k^n;\, n \in \mathbb{N}, k \in \mathbb{Z}_+)$ of $(\mathcal{F}_t^B)$-stopping times such that $\tau_0^n = 0$, $\tau_k^n \leq \tau_{k+1}^n$,*

$$\mathrm{Law}\big(B_{\tau_k^n};\, k \in \mathbb{Z}_+\big) = \mathrm{Law}\big(X_{k/n}^n;\, k \in \mathbb{Z}_+\big), \quad n \in \mathbb{N} \tag{2.4}$$

and

$$\forall m \in \mathbb{N}, \quad \max_{k=0,\dots,mn} \Big|\tau_k^n - \frac{k}{n}\Big| \xrightarrow[n\to\infty]{\mathsf{P}} 0. \tag{2.5}$$

Proof. Let $(W_t;\, t \geq 0)$ be a Brownian motion started at zero. According to Proposition 2.5, there exists a stopping time τ such that $\mathsf{E}\tau = 1$ and $\mathrm{Law}(W_\tau) = \mathrm{Law}(\xi_k)$. The random variable τ is a functional of the paths of W, which will be denoted as $\tau = \tau(W)$. Let us now construct a sequence of processes $(W_t^n;\, t \geq 0)$ and a sequence of random variables (τ_n) by the following procedure:

$$\begin{aligned} W_t^1 &= W_t, & \tau_1 &= \tau(W^1), \\ W_t^2 &= W_{t+\tau_1}^1 - W_{\tau_1}^1, & \tau_2 &= \tau(W^2), \\ W_t^3 &= W_{t+\tau_2}^2 - W_{\tau_2}^2, & \tau_3 &= \tau(W^3)\dots \end{aligned}$$

Note that each τ_k is a functional of W. This will be denoted as $\tau_k = \tau_k(W)$.

It follows from the strong Markov property of W that the sequence $(\tau_n)_{n=1}^{\infty}$ is a sequence of i.i.d. random variables with $\mathsf{E}\tau_n = 1$. By the strong law of large numbers, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\mathsf{P}\left\{\forall n \geq N_1,\ \Big|\sum_{i=1}^{n} \tau_i - n\Big| < n\varepsilon\right\} > 1 - \varepsilon.$$

Take $N_2 \in \mathbb{N}$ such that

$$\mathsf{P}\left\{\frac{1}{N_2}\sum_{i=1}^{N_1} \tau_i < \frac{\varepsilon}{2}\right\} > 1 - \varepsilon, \qquad \frac{N_1}{N_2} < \frac{\varepsilon}{2}.$$

Then, for any $n \geq N_2$, $m \in \mathbb{N}$, we have

$$\mathsf{P}\left\{\forall k = 1, \dots mn,\ \Big|\frac{1}{n}\sum_{i=1}^{k} \tau_i - \frac{k}{n}\Big| < m\varepsilon\right\} > 1 - 2\varepsilon.$$

For any $n \in \mathbb{N}$, the process $B_t^n = \sqrt{n} B_{t/n}$ is again a Brownian motion started at zero. Therefore, the random variables

$$\tau_k^n = \frac{1}{n}\sum_{i=1}^{k} \tau_i(B^n), \quad k \in \mathbb{Z}_+$$

are correctly defined. One can easily verify that they satisfy the conditions of the lemma. □

Definition 2.7. A sequence of d-dimensional processes $(Z_t^n;\, t \geq 0)$ converges to a process $(Z_t;\, t \geq 0)$ *in probability uniformly on compact intervals* if

$$\forall t \geq 0,\ \sup_{s \leq t} \|Z_s^n - Z_s\| \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

We will use the notation:

$$(Z_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{u.p.}} (Z_t;\, t \geq 0).$$

Lemma 2.8. *Let f be piecewise continuous. Let $(B_t;\, t \geq 0)$ be a Brownian motion started at zero and $(\tau_k^n;\, n \in \mathbb{N}, k \in \mathbb{Z}_+)$ be the collection of stopping times given by Lemma 2.6. Set $\xi_k^n = B_{\tau_k^n} - B_{\tau_{k-1}^n}$ and define the processes $(X_t^n;\, t \geq 0)$, $(Y_t^n;\, t \geq 0)$ through ξ_k^n using (2.1), (2.2). Then*

$$(X_t^n, Y_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{u.p.}} \left(B_t, \int_0^t f(B_s)\,dB_s;\, t \geq 0\right).$$

Proof. It follows from the continuity of B that

$$\forall t \geq 0, \quad \sup_{\{x,y\in[0,t]\,:\,|x-y|<\varepsilon\}} |B_y - B_x| \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} 0. \tag{2.6}$$

Furthermore, (2.5) implies that

$$\forall t \geq 0, \quad \max_{\{k\,:\,\tau_k^n\in[0,t]\}} |\tau_k^n - \tau_{k-1}^n| \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

Since the processes X^n and B coincide at the times τ_k^n and the process X^n is linear on each $[\tau_{k-1}^n, \tau_k^n]$, we arrive at

$$(X_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{u.p.}} (B_t;\, t \geq 0).$$

Thus, it only remains to prove the convergence

$$(Y_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{u.p.}} \biggl(\int_0^t f(B_s)dB_s;\, t \geq 0\biggr). \tag{2.7}$$

It will suffice to check (2.7) for one-dimensional functions f. We will do this in several steps.

Step 1. Suppose that $f(x) = I(x > 0)$. Let us consider the processes

$$\widetilde{Y}_t^n = \int_0^t H_s^n dB_s, \quad t \geq 0, \tag{2.8}$$

where

$$H_t^n = \sum_{i=1}^{\infty} I(\tau_{i-1}^n < t \leq \tau_i^n) f(B_{\tau_{i-1}^n}), \quad t \geq 0. \tag{2.9}$$

It follows from the equality

$$Y_{\tau_k^n}^n = \sum_{i=1}^{k} f\bigl(B_{\tau_{i-1}^n}\bigr)\bigl(B_{\tau_i^n} - B_{\tau_{i-1}^n}\bigr), \quad k \in \mathbb{Z}_+$$

that $\widetilde{Y}_{\tau_k^n}^n = Y_{\tau_k^n}^n$ for any $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$.

Fix $t \geq 0$. It follows from (2.6) that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in [\varepsilon, t]$ with $|y - x| < \delta$, we have

$$\mathsf{P}\bigl\{f(B) \text{ is constant on } [x,y]\bigr\} > 1 - \varepsilon.$$

Combining this with (2.5), we deduce that there exists $N \in \mathbb{N}$ such that, for any $n \geq N$ and any k between εn and tn,

$$\mathsf{P}\bigl\{f(B) \text{ is constant on } [\tau_{k-1}^n, \tau_k^n]\bigr\} > 1 - \varepsilon.$$

Consequently,

$$\forall t \geq 0, \quad \mathsf{E}\int_0^t (H_s^n - f(B_s))^2 ds \xrightarrow[n\to\infty]{} 0.$$

Due to the Burkholder-Davis-Gundy inequality (see [10; Ch. IV, (4.1)]) or rather Doob's L^2-inequality,

$$\forall t \geq 0, \quad \mathsf{E}\sup_{s\leq t}\biggl(\widetilde{Y}_s^n - \int_0^s f(B_u)dB_u\biggr)^2 \xrightarrow[n\to\infty]{} 0.$$

In particular,

$$\forall t \geq 0, \quad \max_{\{k:\tau_k^n \in [0,t]\}} \left| \widetilde{Y}^n_{\tau_k^n} - \int_0^{\tau_k^n} f(B_s)dB_s \right| \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

Using the equality $\widetilde{Y}^n_{\tau_k^n} = Y^n_{\tau_k^n}$ and keeping in mind that Y^n is linear on each $[\tau_{k-1}^n, \tau_k^n]$, we get (2.7).

Step 2. The same arguments as above show that (2.7) holds for $f(x) = I(x > a)$ and $f(x) = I(x < a)$, where $a \in \mathbb{R}$. Since (2.7) holds for $f = 1$, it also holds for $f(x) = I(x \geq a)$ and $f(x) = I(x \leq a)$, where $a \in \mathbb{R}$.

Step 3. By linearity, we extend (2.7) to the functions of the form $f(x) = \sum_{i=1}^m \lambda_i I(x \in J_i)$, where J_i are intervals (that may be closed, open or semi-open).

Step 4. Let f be a piecewise continuous function with compact support. Then f can be uniformly approximated by a sequence of functions $(f_m)_{m=1}^\infty$ of the form described in Step 3. Define $\widetilde{Y}^n$, H^n by (2.8), (2.9) and define $\widetilde{Y}^{nm}$, H^{nm} in the same way with f replaced by f_m. Then

$$\forall t \geq 0, \ \ |H_t^{nm} - H_t^n| \leq \sup_{x\in\mathbb{R}} |f^m(x) - f(x)| \xrightarrow[m\to\infty]{} 0.$$

Consequently,

$$\forall t \geq 0, \ \ \sup_{s\leq t} \left| \int_0^s H_u^{nm} dB_u - \int_0^s H_u^n dB_u \right| \xrightarrow[m\to\infty]{\mathsf{P}} 0,$$

and the convergence is uniform in n. Thus,

$$\forall t \geq 0, \quad \max_{\{k\,:\,\tau_k^n \in [0,t]\}} \left| \widetilde{Y}^{nm}_{\tau_k^n} - \widetilde{Y}^n_{\tau_k^n} \right| \xrightarrow[m\to\infty]{\mathsf{P}} 0,$$

and the convergence is uniform in n. Combining this with the properties

$$\forall t \geq 0, \ \forall m \in \mathbb{N}, \quad \max_{\{k\,:\,\tau_k^n \in [0,t]\}} \left| \widetilde{Y}^{nm}_{\tau_k^n} - \int_0^{\tau_k^n} f^m(B_s)dB_s \right| \xrightarrow[n\to\infty]{\mathsf{P}} 0$$

and

$$\forall t \geq 0, \quad \max_{\{k\,:\,\tau_k^n \in [0,t]\}} \left| \int_0^{\tau_k^n} f^m(B_s)dB_s - \int_0^{\tau_k^n} f(B_s)dB_s \right| \xrightarrow[m\to\infty]{\mathsf{P}} 0,$$

we conclude that

$$\forall t \geq 0, \quad \max_{\{k\,:\,\tau_k^n \in [0,t]\}} \left| \widetilde{Y}^n_{\tau_k^n} - \int_0^{\tau_k^n} f(B_s)dB_s \right| \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

Using the equality $\widetilde{Y}^n_{\tau_k^n} = Y^n_{\tau_k^n}$ and keeping in mind that Y^n is linear on each $[\tau_{k-1}^n, \tau_k^n]$, we get (2.7).

Step 5. Let f be piecewise continuous. Then there exists a sequence $(f_m)_{m=1}^\infty$ of piecewise continuous functions with compact support such that $f_m = f$ on $[-m, m]$. Define Y^{nm} in the same way as Y^n with f replaced by f_m. Fix $t \geq 0$. On the set $A_m = \{\forall s \leq t, \, |B_s| \leq m\}$, we have $Y_s^{nm} = Y_s^n$, $s \leq t$. Furthermore, $\mathsf{P}(A_m) \xrightarrow[m\to\infty]{} 1$. We now proceed similarly to the previous step. □

Proposition 2.9 (Prokhorov). *Let $(\xi_k)_{k=1}^\infty$ be a sequence of i.i.d. random variables with $\mathsf{E}\xi_k = 0$, $\mathsf{E}\xi_k^2 = 1$. Set $S_n = \xi_1 + \dots + \xi_n$. Then the distributions of $\frac{1}{\sqrt{n}} S_n$ converge in variation to the normal distribution $\mathcal{N}(0,1)$ if and only if there exists $m \in \mathbb{N}$ such that the distribution of S_m has an absolutely continuous* (*with respect to the Lebesgue measure*) *component.*

For the proof, see [7; Ch. IV, §4].

Lemma 2.10. *Let f be locally bounded. Suppose that there exists $m \in \mathbb{N}$ such that the distribution of $\xi_1 + \dots + \xi_m$ has an absolutely continuous (with respect to the Lebesgue measure) component. Let $(B_t;\, t \ge 0)$ be a Brownian motion started at zero and $(\tau_k^n;\, n \in \mathbb{N}, k \in \mathbb{Z}_+)$ be the collection of stopping times given by Lemma 2.6. Set $\xi_k^n = B_{\tau_k^n} - B_{\tau_{k-1}^n}$ and define the processes $(X_t^n;\, t \ge 0)$, $(Y_t^n;\, t \ge 0)$ through ξ_k^n by (2.1), (2.2). Then*

$$(X_t^n, Y_t^n;\, t \ge 0) \xrightarrow[n\to\infty]{\text{u.p.}} \left(B_t, \int_0^t f(B_s)dB_s;\, t \ge 0\right).$$

Proof. Similarly to Lemma 2.8, it will suffice to prove only (2.7) for one-dimensional functions f. We will do this in several steps.

Step 1. Let $a \ge 0$. Let us prove (2.7) for the functions f of the form $f(x) = I(x \in A)$, where $A \in \mathcal{B}([-a,a])$. Consider the set

$$\mathcal{M} = \{A \in \mathcal{B}([-a,a]) : (2.7) \text{ holds for } f = I_A\}.$$

It follows from Lemma 2.8 that $\mathcal{M}$ contains all the intervals in $[-a,a]$. Let us check that $\mathcal{M}$ is a monotone class, i.e.

i) $[-a,a] \in \mathcal{M}$;

ii) if $A, B \in \mathcal{M}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{M}$;

iii) if $(A_m)_{m=1}^\infty \in \mathcal{M}$ and $A_m \subseteq A_{m+1}$, then $\bigcup_{m=1}^\infty A_m \in \mathcal{M}$.

The only nontrivial point is iii). Set $A = \bigcup_{m=1}^\infty A_m$, $f_m = I_{A_m}$, $f = I_A$. Define $\widetilde{Y}^n$, H^n by (2.8), (2.9) and define $\widetilde{Y}^{nm}$, H^{nm} in the same way with f replaced by f_m. Let μ_L denote the Lebesgue measure on $\mathbb{R}$. It follows from Proposition 2.9 that, for any $\varepsilon > 0$, there exist $N(\varepsilon) \in \mathbb{N}$ and $\delta(\varepsilon) > 0$ such that, for any $D \in \mathcal{B}(\mathbb{R})$ with $\mu_L(D) < \delta(\varepsilon)$ and any $n \ge N(\varepsilon)$, $\mathsf{P}\big(\frac{1}{\sqrt{n}}S_n \in D\big) < \varepsilon$. Then, for any $D \in \mathcal{B}(\mathbb{R})$ with $\mu_L(D) < \sqrt{\varepsilon}\delta(\varepsilon)$, any $n \ge \varepsilon^{-1}N(\varepsilon)$ and any $k \ge \varepsilon n$, we have $\mathsf{P}\big(\frac{1}{\sqrt{n}}S_k \in D\big) < \varepsilon$. Find $M(\varepsilon) \in \mathbb{N}$ such that $\mu_L\big(A \setminus A_{M(\varepsilon)}\big) < \sqrt{\varepsilon}\delta(\varepsilon)$. In view of the equality $\mathrm{Law}\big(\frac{1}{\sqrt{n}}S_k\big) = \mathrm{Law}(B_{\tau_k^n})$, for any $n \ge \varepsilon^{-1}N(\varepsilon)$, $k \ge \varepsilon n$, $m \ge M(\varepsilon)$, we get

$$\mathsf{P}\big(B_{\tau_k^n} \in A \setminus A_m\big) < \varepsilon.$$

Fix $t \ge 0$. For any $n \ge \varepsilon^{-1}N(\varepsilon)$, $m \ge M(\varepsilon)$, we have

$$\begin{aligned}
&\mathsf{E}\int_{\tau_{[\varepsilon n]+1}^n}^{\tau_{[tn]}^n} (H_s^{nm} - H_s^n)^2 ds \\
&= \sum_{k=[\varepsilon n]+1}^{[tn]-1} \mathsf{E}\big(H_{\tau_k^n}^{nm} - H_{\tau_k^n}^n\big)^2(\tau_{k+1}^n - \tau_k^n) \\
&= \sum_{k=[\varepsilon n]+1}^{[tn]-1} \mathsf{E}\big[\mathsf{E}\big[\big(H_{\tau_k^n}^{nm} - H_{\tau_k^n}^n\big)^2(\tau_{k+1}^n - \tau_k^n) \mid \mathcal{F}_{\tau_k^n}^B\big]\big] \\
&= \sum_{k=[\varepsilon n]+1}^{[tn]-1} \frac{1}{n}\mathsf{E}\big(H_{\tau_k^n}^{nm} - H_{\tau_k^n}^n\big)^2 \\
&= \frac{1}{n}\sum_{k=[\varepsilon n]+1}^{[tn]-1} \mathsf{P}(B_{\tau_{k-1}^n} \in A \setminus A_m) \le t\varepsilon.
\end{aligned}$$

Moreover,

$$\mathsf{E}\int_0^{\tau_{[\varepsilon n]+1}}(H_s^{nm}-H_s^n)^2ds \le \mathsf{E}\tau_{[\varepsilon n]+1}^n < \frac{\varepsilon n+1}{n}.$$

It follows from the Burkholder-Davis-Gundy inequality (see [10; Ch. IV, (4.1)]) or rather Doob's L^2-inequality that

$$\mathsf{E}\sup_{s\le\tau_{[tn]}^n}(\widetilde{Y}_s^{nm}-\widetilde{Y}_s^n)^2 \xrightarrow[n,m\to\infty]{} 0.$$

Arguing in the same way as in the proof of Lemma 2.8 (Step 4), we deduce that

$$(Y_t^n;\, t\ge 0) \xrightarrow[n\to\infty]{\text{u.p.}} \Bigl(\int_0^t f(B_s)dB_s;\, t\ge 0\Bigr),$$

which means that $A\in\mathcal{M}$.

Applying the monotone class lemma (see [10; Ch. 0, (2.1)]), we conclude that $\mathcal{M}=\mathcal{B}([-a,a])$.

Step 2. By linearity, we extend (2.7) to the functions of the form $f(x)=\sum_{i=1}^m \lambda_i I(x\in A_i)$, where $A_i\in\mathcal{B}([-a,a])$.

Step 3. Let f be a bounded function with compact support. Then f can be uniformly approximated by the functions of the form described in Step 2. Using the same arguments as in the proof of Lemma 2.8 (Step 4), we get (2.7) for f.

Step 4. Similar arguments as in the proof of Lemma 2.8 (Step 5) show that (2.7) is true for any locally bounded f. □

`Proof of Theorem 2.2.` In view of (2.4), the process $(X_t^n, Y_t^n;\, t\ge 0)$ defined in Lemma 2.8 has the same distribution as the "original" process $(X_t^n, Y_t^n;\, t\ge 0)$ that appears in (2.3). The desired result now follows from the fact that the convergence in probability uniformly on compact intervals implies the weak convergence. □

`Proof of Theorem 2.3.` This theorem is proved in the same way as Theorem 2.2 (with Lemma 2.8 replaced by Lemma 2.10). □

3 Approximation of the Brownian Local Time

1. Definitions and known facts. Let $(Z_t;\, t\ge 0)$ be a continuous semimartingale.

Definition 3.1. The *local time of* Z *at a point* $a\in\mathbb{R}$ is the random process $(L_t^a(Z);\, t\ge 0)$ that satisfies the equality

$$|Z_t-a| = |Z_0-a| + \int_0^t \operatorname{sgn}(Z_s-a)dZ_s + L_t^a(Z), \quad t\ge 0. \tag{3.1}$$

Formula (3.1) is called the *Tanaka formula.*

`Remark.` The value $\operatorname{sgn} 0$ is taken to be equal to -1. If Z is a local martingale, then the value $\operatorname{sgn} 0$ is not important since in this case

$$\int_0^t I(Z_s=a)dZ_s = 0, \quad t\ge 0.$$

□

Proposition 3.2 (Itô-Tanaka formula). *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a difference of two convex functions. Then*

$$\varphi(Z_t) = \varphi(Z_0) + \int_0^t \varphi'_-(Z_s)dZ_s + \frac{1}{2}\int_{\mathbb{R}} L_t^z(Z)\varphi''(dz), \quad t \geq 0,$$

where φ'_- denotes the left-hand derivative of φ and φ'' denotes the second derivative of φ (this is a signed measure on $\mathbb{R}$).

For the proof, see [10; Ch. VI, (1.5)].

Proposition 3.3. *The process $(L_t^a(Z);\, t \geq 0)$ is an increasing continuous process and the measure $dL_t^a(Z)$ is a.s. carried by the set $\{t \geq 0 : Z_t = a\}$.*

For the proof, see [10; Ch. VI, (1.2), (1.3)].

More information on the local time as well as other equivalent definitions of this process can be found in [10; Ch. VI], [8; Ch. 2].

2. The results. Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathsf{P}(\xi_k = 1) = \mathsf{P}(\xi_k = -1) = \frac{1}{2}$. Let us set

$$X_k = \sum_{i=1}^{k} \xi_i, \qquad k \in \mathbb{Z}_+,$$
$$L_k = \sum_{i=0}^{k-1} I(X_i = 0), \quad k \in \mathbb{Z}_+.$$

For each $n \in \mathbb{N}$, we consider

$$X_{k/n}^n = \frac{1}{\sqrt{n}} X_k, \quad L_{k/n}^n = \frac{1}{\sqrt{n}} L_k, \quad k \in \mathbb{Z}_+$$

and construct the processes $(X_t^n;\, t \geq 0)$, $(L_t^n;\, t \geq 0)$ by linear interpolation of $(X_{k/n}^n;\, k \in \mathbb{Z}_+)$, $(L_{k/n}^n;\, k \in \mathbb{Z}_+)$.

Theorem 3.4. *We have*

$$(X_t^n, L_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{Law}} (B_t, L_t;\, t \geq 0),$$

where B is a Brownian motion started at zero and L is its local time at zero.

`Proof`. Set

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

For each $n \in \mathbb{N}$, consider

$$Y_{k/n}^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} f\big(X_{(i-1)/n}^n\big)\xi_i, \quad k \in \mathbb{Z}_+.$$

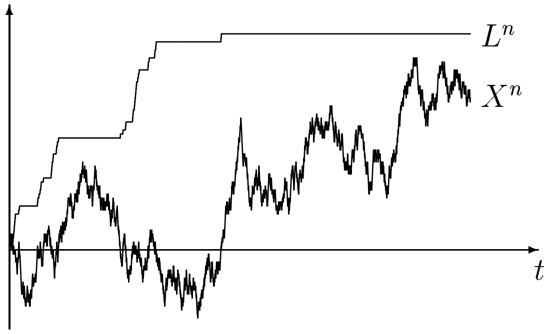


Figure 2. Simulated paths of X^n and L^n $(n = 2500)$

Comparing this expression with the equalities

$$X^n_{k/n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \xi_i, \qquad k \in \mathbb{Z}_+,$$
$$L^n_{k/n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} I\big(X^n_{(i-1)/n} = 0\big), \quad k \in \mathbb{Z}_+,$$

one can easily check that

$$L^n_{k/n} = |X^n_{k/n}| - Y^n_{k/n}, \quad k \in \mathbb{Z}_+.$$

Consequently,

$$L^n_t = |X^n_t| - Y^n_t, \quad t \ge 0.$$

Now, the result follows from Theorem 2.2 and the equality

$$L_t = |B_t| - \int_0^t f(B_s) dB_s, \quad t \ge 0$$

(see (3.1)). □

4 Approximations of Skew Brownian Motions

1. Definitions and known facts. Let $(B_t;\, t \ge 0)$ be a Brownian motion started at a point B_0. Let $p \in [0,1]$. Set

$$A_t = \int_0^t (p^2 I(B_s \ge 0) + (1-p)^2 I(B_s < 0)) ds, \quad t \ge 0, \tag{4.1}$$
$$\tau_t = \inf\{s \ge 0 : A_s > t\}, \quad t \ge 0, \tag{4.2}$$
$$M_t = B_{\tau_t}, \quad t \ge 0, \tag{4.3}$$
$$B^p_t = \varphi(M_t), \quad t \ge 0, \tag{4.4}$$

where

$$\varphi(x) = \begin{cases} px & \text{if } x \ge 0, \\ (1-p)x & \text{if } x < 0. \end{cases} \tag{4.5}$$

Definition 4.1. The process $(B_t^p;\, t \ge 0)$ is called a *skew Brownian motion with parameter p* started at B_0^p.

`Remarks.` (i) A skew Brownian motion with parameter 1 coincides in distribution with the modulus of a Brownian motion (see [8; §2.11]). A skew Brownian motion with parameter $\frac{1}{2}$ is an ordinary Brownian motion. A skew Brownian motion with parameter 0 coincides in distribution with the modulus of a Brownian motion multiplied by -1.

(ii) The construction (4.1)–(4.5) is precisely Feller's construction of a diffusion from a Brownian motion through a time change and a space transformation with $\varphi : \mathbb{R} \to \mathbb{R}$ and $A_t = \int_0^t (\varphi'(B_s))^2 ds$.

(iii) There exist other possible ways of defining a skew Brownian motion. This process may be defined as a Markov process with a given transition function (see [10; Ch. III, (1.16)]); it may be defined as a diffusion process with a given generator (see [10; Ch. VII, (1.23)]); it may be defined as the solution of a stochastic differential equation with a generalized drift (see [6]). One of the most transparent ways of defining the skew Brownian motion is based on the excursion theory. Informally, a skew Brownian motion with parameter p is obtained from the modulus of a Brownian motion by changing the sign of each of its excursions with probability $1-p$. □

Lemma 4.2. *Let $(B_t^p;\, t \ge 0)$ be a skew Brownian motion with parameter p started at a point B_0^p.*

(i) *The process B^p has the strong Markov property.*

(ii) *The process B^p is a semimartingale.*

(iii) *We have* $\mathrm{Law}(|B_t^p|;\, t \ge 0) = \mathrm{Law}(|B_t|;\, t \ge 0)$, *where B is a Brownian motion started at B_0^p.*

(iv) *Let $a < B_0^p < c$ and set $T_a(B^p) = \inf\{t \ge 0 : B_t^p = a\}$, $T_c(B^p) = \inf\{t \ge 0 : B_t^p = c\}$. Then*

$$\mathsf{P}\{T_c(B^p) < T_a(B^p)\} = \frac{\varphi^{-1}(B_0^p) - \varphi^{-1}(a)}{\varphi^{-1}(c) - \varphi^{-1}(a)}.$$

(*If $p = 1$, we consider only $a \ge 0$; if $p = 0$, we consider only $c \le 0$.*)

`Proof.` **(i)** If $p = 1$, then B^p is the modulus of a Brownian motion (see [8; §2.11]), and this process has the strong Markov property (see [10; Ch. XI, §1]). For $p \in (0,1)$, statement (i) follows from [10; Ch. X, (2.18)].

(ii) If $p = 1$, then B^p is the modulus of a Brownian motion. It follows from the Tanaka formula (3.1) that this process is a semimartingale. For $p \in (0,1)$, the process M given by (4.3) is a local martingale (see [10; Ch. V, (1.5)]). It follows from the Itô-Tanaka formula (Proposition 3.2) that

$$\varphi(M_t) = \varphi(M_0) + \int_0^t \varphi'_-(M_s)dM_s + \frac{2p-1}{2}L_t^0(M), \quad t \ge 0. \tag{4.6}$$

Hence, $\varphi(M)$ is a semimartingale.

(iii) For $p = 0, 1$, this statement follows from [8; §2.11]. For $p \in (0,1)$, this statement follows from [10; Ch. XII, (2.16)].

(iv) This statement is a consequence of the following fact. Let $(B_t;\, t \ge 0)$ be a Brownian motion started at a point B_0. Let $a < B_0 < c$. Set $T_a(B) = \inf\{t \ge 0 : B_t = a\}$, $T_c(B) = \inf\{t \ge 0 : B_t = c\}$. Then

$$\mathsf{P}\{T_c(B) < T_a(B)\} = \frac{B_0 - a}{c - a}$$

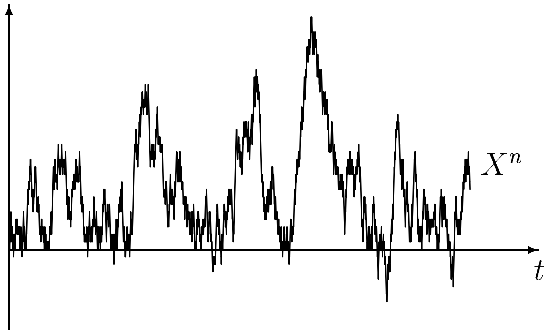


Figure 3. A simulated path of X^n $(p = 0.9,\ n = 2500)$

(see [10; Ch. II, (3.8)]. □

Remark. As opposed to the case of a Brownian motion, the stochastic integral

$$\int_0^t I(B_s^p = 0)dB_s^p$$

is not equal to zero. This follows from (4.6). □

For more information on skew Brownian motions, see [1], [2], [6].

2. The results. Let $(X_k;\, k \in \mathbb{Z}_+)$ be an integer-valued Markov chain with $X_0 = 0$ and the transition probabilities

$$\mathsf{P}(X_{k+1} = i+1 \mid X_k = i) = \frac{1}{2}, \quad \mathsf{P}(X_{k+1} = i-1 \mid X_k = i) = \frac{1}{2} \quad \text{if } i \neq 0,$$
$$\mathsf{P}(X_{k+1} = 1 \mid X_k = 0) = p, \qquad \mathsf{P}(X_{k+1} = -1 \mid X_k = 0) = 1 - p,$$

where $p \in [0,1]$. Let $f : \mathbb{R} \to \mathbb{R}^d$ be a Borel function. For each $n \in \mathbb{N}$, we consider

$$X_{k/n}^n = \frac{1}{\sqrt{n}} X_k, \qquad k \in \mathbb{Z}_+, \tag{4.7}$$

$$Y_{k/n}^n = \sum_{i=1}^k f\bigl(X_{(i-1)/n}^n\bigr)\bigl(X_{i/n}^n - X_{(i-1)/n}^n\bigr), \quad k \in \mathbb{Z}_+ \tag{4.8}$$

and construct the processes $(X_t^n;\, t \geq 0)$, $(Y_t^n;\, t \geq 0)$ by linear interpolation of $(X_{k/n}^n;\, k \in \mathbb{Z}_+)$, $(Y_{k/n}^n;\, k \in \mathbb{Z}_+)$.

Theorem 4.3. *If f is piecewise continuous, then*

$$(X_t^n, Y_t^n;\, t \geq 0) \xrightarrow[n\to\infty]{\text{Law}} \left(B_t^p, \int_0^t f(B_s^p)dB_s^p;\, t \geq 0\right), \tag{4.9}$$

where B^p is a skew Brownian motion with parameter p started at zero.

Remark. If f is only locally bounded, then the conclusion of Theorem 4.3 does not hold. In order to see this, one needs only to consider the function given by Example 2.4. □

3. The proofs. Theorem 4.3 follows from Lemma 4.6 given below.

Lemma 4.4. *Let $(B_t^p;\ t\ge 0)$ be a skew Brownian motion with parameter $p\in[0,1]$ started at a point B_0^p. Let $a\ne 0$ and $(H_t;\ t\ge 0)$ be a bounded $\left(\mathcal{F}_t^{B^p}\right)$-predictable process such that, for any $t\ge 0$, H_t equals zero on the set $\{B_t^p\ne a\}$. Then*

$$\int_0^t H_s dB_s^p = 0,\quad t\ge 0.$$

Proof. By the Itô-Tanaka formula,

$$B_t^p = B_0^p + \int_0^t \varphi'_-(M_s)dM_s + \frac{2p-1}{2}L_t^0(M),\quad t\ge 0,$$

where M is given by (4.3) and φ is given by (4.5). Hence,

$$\begin{aligned}\int_0^t H_s dB_s^p &= \int_0^t H_s\varphi'_-(M_s)dM_s + \frac{2p-1}{2}\int_0^t H_s dL_s^0(M)\\ &= \int_0^t H_s\varphi'_-(a)I(M_s=a)dM_s = N_t,\quad t\ge 0.\end{aligned}$$

(In the latter equality we applied Proposition 3.3). The process N is a local martingale and

$$\langle N\rangle_t = \int_0^t (H_s\varphi'_-(a)I(M_s=a))^2 d\langle M\rangle_s,\quad t\ge 0.$$

By the occupation times formula (see [10; Ch. VI, (1.6)]),

$$\int_0^t I(M_s=a)d\langle M\rangle_s = \int_{\mathbb{R}} I(x=a)L_t^x(M)dx = 0,\quad t\ge 0.$$

Hence, $N=0$. This is the desired statement. □

Lemma 4.5. *Let $(B_t^p;\ t\ge 0)$ be a skew Brownian motion with parameter p started at zero. Define a collection of stopping times $(\tau_k^n;\ n\in\mathbb{N},k\in\mathbb{Z}_+)$ by $\tau_0^n=0$,*

$$\tau_{k+1}^n = \inf\left\{t\ge\tau_k^n : \left|B_t^p - B_{\tau_k^n}^p\right| \ge \frac{1}{\sqrt{n}}\right\}.$$

Then

$$\operatorname{Law}\left(B_{\tau_k^n}^p;\ k\in\mathbb{Z}_+\right) = \operatorname{Law}(X_{k/n}^n;\ k\in\mathbb{Z}_+) \tag{4.10}$$

and

$$\forall m\in\mathbb{N},\quad \max_{k=0,\dots,mn}\left|\tau_k^n - \frac{k}{n}\right| \xrightarrow[n\to\infty]{\mathsf{P}} 0. \tag{4.11}$$

Proof. Since B^p is a strong Markov process (see Lemma 4.2 (i)), the sequence $(B_{\tau_k^n}^p;\ k\in\mathbb{Z}_+)$ is a Markov chain. It follows from Lemma 4.2 (iv) that this Markov chain has the same transition probabilities as $(X_{k/n}^n;\ k\in\mathbb{Z}_+)$. Thus, we get (4.10).

In order to prove (4.11), let us consider a Brownian motion $\widetilde{B}$ started at zero and a collection of stopping times $(\widetilde{\tau}_k^n;\ n\in\mathbb{N},k\in\mathbb{Z}_+)$ given by $\widetilde{\tau}_0^n=0$,

$$\widetilde{\tau}_{k+1}^n = \inf\left\{t\ge\widetilde{\tau}_k^n : \left|\widetilde{B}_t - \widetilde{B}_{\tau_k^n}\right| \ge \frac{1}{\sqrt{n}}\right\}.$$

Note that we can write

$$\tau_{k+1}^n = \inf\left\{ t \ge \tau_k^n : \big||B_t^p| - |B_{\tau_k^n}^p|\big| \ge \frac{1}{\sqrt{n}} \right\}$$

and

$$\widetilde{\tau}_{k+1}^n = \inf\left\{ t \ge \widetilde{\tau}_k^n : \big||\widetilde{B}_t| - |\widetilde{B}_{\tau_k^n}|\big| \ge \frac{1}{\sqrt{n}} \right\}.$$

As $\mathrm{Law}(|B_t^p|;\, t \ge 0) = \mathrm{Law}(|\widetilde{B}_t|;\, t \ge 0)$ (see Lemma 4.2 (iii)), we get $\mathrm{Law}(\tau_k^n;\, k \in \mathbb{Z}_+) = \mathrm{Law}(\widetilde{\tau}_k^n;\, k \in \mathbb{Z}_+)$. Now, (4.11) follows from the equality $\mathsf{E}(\widetilde{\tau}_{k+1}^n - \widetilde{\tau}_k^n) = \frac{1}{n}$ combined with the arguments used in the proof of Lemma 2.6. □

Lemma 4.6. *Let f be piecewise continuous. Let $(B_t^p;\, t \ge 0)$ be a skew Brownian motion with parameter p started at zero. Let $(\tau_k^n;\, n \in \mathbb{N}, k \in \mathbb{Z}_+)$ be the collection of stopping times given by Lemma 4.5. Set $X_{k/n}^n = B_{\tau_k^n}^p$ and define Y^n through X^n using* (4.8). *Then*

$$(X_t^n, Y_t^n;\, t \ge 0) \xrightarrow[n\to\infty]{\text{u.p.}} \left(B_t^p, \int_0^t f(B_s^p)\,dB_s^p;\, t \ge 0\right).$$

Proof. As in Lemma 2.8, it will suffice to prove the convergence

$$(Y_t^n;\, t \ge 0) \xrightarrow[n\to\infty]{\text{u.p.}} \left(\int_0^t f(B_s^p)\,dB_s^p;\, t \ge 0\right) \tag{4.12}$$

for one-dimensional functions f. We will do this in several steps.

Step 1. Suppose that $f(x) = I(x > 0)$. Consider the processes

$$\widetilde{Y}_t^n = \int_0^t H_s^n\,dB_s^p, \quad t \ge 0, \tag{4.13}$$

where

$$H_t^n = \sum_{i=1}^{\infty} I(\tau_{i-1}^n \le t < \tau_i^n) f\big(B_{\tau_{i-1}^n}^p\big), \quad t \ge 0.$$

It is easy to see that H_t^n equals $f(B_t^p)$ on the set $\big\{B_t^p \notin \big(-\frac{1}{\sqrt{n}}, 0\big) \cup \big(0, \frac{1}{\sqrt{n}}\big)\big\}$. Hence, the processes H^n tend to $f(B^p)$ pointwise. Now, it follows from the Lebesgue dominated convergence theorem for stochastic integrals (see [9; Ch. I, (4.40)]) that

$$(\widetilde{Y}_t^n;\, t \ge 0) \xrightarrow[n\to\infty]{\text{u.p.}} \left(\int_0^t f(B_s^p)\,dB_s^p;\, t \ge 0\right).$$

Using the equality $\widetilde{Y}_{\tau_k^n}^n = Y_{\tau_k^n}^n$ and keeping in mind that Y^n is linear on each $[\tau_{k-1}^n, \tau_k^n]$, we get (4.12).

Step 2. Let $f(x) = I(x > a)$, where $a \ne 0$. Let $\widetilde{Y}^n$ be the process defined by (4.13). In view of Lemma 4.4, $\widetilde{Y}^n$ can be rewritten as

$$\widetilde{Y}_t^n = \int_0^t K_s^n\,dB_s^p, \quad t \ge 0,$$

where $K_t^n = H_t^n I(B_t^p \neq a)$. It is easy to see that K_t^n equals $f(B_t^p)$ on the set $\{B_t^p \notin (a - \frac{1}{\sqrt{n}}, a) \cup (a, a + \frac{1}{\sqrt{n}})\}$. Hence, the processes H^n tend to $f(B^p)$ pointwise. We now proceed similarly to Step 1.

Step 3. We derive (4.12) for piecewise continuous functions in the same way as in Lemma 2.8 (Steps 2–5). □

`Proof of Theorem 4.3.` In view of (4.10), the process $(X_t^n, Y_t^n;\, t \geq 0)$ defined in Lemma 4.6 has the same distribution as the "original" process $(X_t^n, Y_t^n;\, t \geq 0)$ that appears in (4.9). The desired result now follows from the fact that the convergence in probability uniformly on compact intervals implies the weak convergence. □

5 Limit Behaviour of the "Horizontal-Vertical" Random Walk

1. Limit behaviour. Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathsf{E}\xi_k = 0$, $\mathsf{E}\xi_k^2 = 1$. Let us set $X_0 = 0$, $Y_0 = 0$,

$$X_{k+1} = \begin{cases} X_k + \xi_{k+1} & \text{if } Y_k > X_k, \\ X_k & \text{if } Y_k \leq X_k, \end{cases}$$

$$Y_{k+1} = \begin{cases} Y_k & \text{if } Y_k > X_k, \\ Y_k - \xi_{k+1} & \text{if } Y_k \leq X_k. \end{cases}$$

For each $n \in \mathbb{N}$, we consider

$$X_{k/n}^n = \frac{1}{\sqrt{n}} X_k, \quad Y_{k/n}^n = \frac{1}{\sqrt{n}} Y_k, \quad k \in \mathbb{Z}_+$$

and construct the processes $(X_t^n;\, t \geq 0)$, $(Y_t^n;\, t \geq 0)$ by linear interpolation of $(X_{k/n}^n;\, k \in \mathbb{Z}_+)$, $(Y_{k/n}^n;\, k \in \mathbb{Z}_+)$.

Theorem 5.1. *Let B be a Brownian motion started at zero. Set*

$$X_t^{\infty} = \int_0^t I(B_s \leq 0) dB_s = \frac{1}{2} L_t - B_t^-, \quad t \geq 0, \tag{5.1}$$

$$Y_t^{\infty} = -\int_0^t I(B_s > 0) dB_s = \frac{1}{2} L_t - B_t^+, \quad t \geq 0. \tag{5.2}$$

Here, L is the local time of B at zero and $B_t^- = -(B_t \wedge 0)$. Then

$$(X_t^n, Y_t^n;\, t \geq 0) \xrightarrow[n \to \infty]{\text{Law}} (X_t^{\infty}, Y_t^{\infty};\, t \geq 0).$$

`Proof.` Let us consider the processes

$$\widetilde{X}_t^n = X_t^n - Y_t^n, \qquad \widetilde{Y}_t^n = X_t^n + Y_t^n.$$

Then

$$\widetilde{X}_{(k+1)/n}^n = \widetilde{X}_{k/n}^n + \frac{\xi_{k+1}}{\sqrt{n}},$$

$$\widetilde{Y}_{(k+1)/n}^n = \widetilde{Y}_{k/n}^n - \frac{\xi_{k+1}}{\sqrt{n}} \operatorname{sgn} \widetilde{X}_{k/n}^n.$$

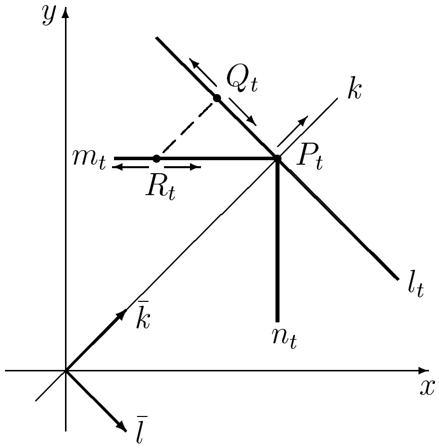


Figure 4. A construction of the limit process

In other words,

$$\widetilde{X}^n_{k/n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{k}\xi_i, \qquad k \in \mathbb{Z}_+$$

$$\widetilde{Y}^n_{k/n} = -\frac{1}{\sqrt{n}}\sum_{i=1}^{k}\xi_i \operatorname{sgn} \widetilde{X}^n_{(i-1)/n}, \quad k \in \mathbb{Z}_+.$$

The result now follows from Theorem 2.2. □

2. Another construction of the limit process. Let us now present another equivalent construction of the limit process $(X^\infty_t, Y^\infty_t;\, t \ge 0)$ given by Theorem 5.1.

Imagine that the line $k = \{x = y\}$ is a string and there is a frame that consists of a line l that is orthogonal to k, a horizontal ray m and a vertical ray n. The elements l, m, n have a common point P. The frame is not motionless, i.e. the point P, to which l, m and n are "attached", can slide along k.

Let $(B_t;\, t \ge 0)$ be a Brownian motion started at zero. Suppose that the point P slides along k in such a way that $P_t = \bar{k}L_t$, where $\bar{k} = (\frac{1}{2}, \frac{1}{2})$ and $(L_t;\, t \ge 0)$ is the local time of B at zero. Then l, m and n also move. We will denote them by l_t, m_t and n_t. Suppose that there is a point $Q_t \in l_t$ given by $Q_t = P_t + \bar{l}B_t$, where $\bar{l} = (\frac{1}{2}, -\frac{1}{2})$. Let R_t be the projection of Q_t on $m_t \cup n_t$ along k (if $B_t < 0$, then $R_t \in m_t$; if $B_t > 0$, then $R_t \in n_t$). The process $(R_t;\, t \ge 0) = (R^x_t, R^y_t;\, t \ge 0)$ is a two-dimensional random process with

$$R^x_t = \frac{1}{2}L_t + \frac{1}{2}B_t - \frac{1}{2}|B_t|, \quad t \ge 0,$$

$$R^y_t = \frac{1}{2}L_t - \frac{1}{2}B_t - \frac{1}{2}|B_t|, \quad t \ge 0.$$

We see that $R^x = X^\infty$, $R^y = Y^\infty$, where X^∞, Y^∞ are given by (5.1), (5.2). Thus, $(R_t;\, t \ge 0)$ is the limit process given by Theorem 5.1.

Remarks. (i) It is seen from this construction that the limit process $(X^\infty_t, Y^\infty_t;\, t \ge 0)$ is a two-dimensional homogeneous Markov process (note that $(B_t, L_t;\, t \ge 0)$ is the one).

(ii) Let $a > 0$. Consider the stopping time $\tau_a = \inf\{t \geq 0 : (X_t^\infty, Y_t^\infty) = (a, a)\}$. Then

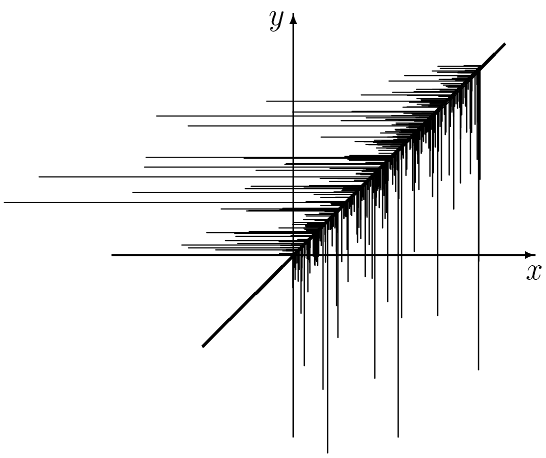
$$\tau_a = \inf\{t \geq 0 : L_t = 2a\} \stackrel{\text{law}}{=} \inf\{t \geq 0 : S_t = 2a\} = \inf\{t \geq 0 : B_t = 2a\}.$$

Here, $S_t = \max_{s \leq t} B_s$, and we applied P. Lévy's theorem (see [10; Ch. VI, (2.3)]). The last random variable is known to have the distribution density

$$p_a(x) = \frac{2a}{\sqrt{2\pi x^3}} e^{-\frac{2a^2}{x}}, \quad x \geq 0$$

(see [10; Ch. III, (3.7)]). In particular, $\mathsf{E}\tau_a = \infty$.

(iii) The above construction of the limit process also shows that the sample paths of (X^∞, Y^∞) (in the phase space) consist of vertical and horizontal intervals. These intervals represent the excursions of the Brownian motion B plotted against its local time at zero (for the definition of an excursion, see [10; Ch. XII]). The length of each (vertical or horizontal) interval shows the height of the corresponding excursion. Thus, the paths of R yield a transparent representation of the excursion process of a Brownian motion. □



`Figure 5.` A simulated path of the limit process

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