DILATATION MONOTONE RISK MEASURES
ARE LAW INVARIANT

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Abstract. We prove that on an atomless probability space every dilatation monotone convex risk measure is law invariant. This result, combined with the known ones, shows the equivalence between dilatation monotonicity and important properties of convex risk measures such as law invariance and second-order stochastic monotonicity.

Key words and phrases. Coherent risk measures, convex risk measures, second-order stochastic dominance, dilatation monotonicity, factor monotonicity, Fatou property, law invariance.

1 Introduction

In the landmark papers [2], [3], [7] Artzner, Delbaen, Eber, and Heath introduced the notion of a *coherent risk measure*. By the definition, a coherent risk measure is a map $\rho : L^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to \mathbb{R}$ satisfying the properties:

(a) (Subadditivity) $\rho(X + Y) \le \rho(X) + \rho(Y)$;
(b) (Monotonicity) If $X \le Y$, then $\rho(X) \ge \rho(Y)$;
(c) (Positive homogeneity) $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \ge 0$;
(d) (Translation invariance) $\rho(X + m) = \rho(X) - m$ for $m \in \mathbb{R}$.

Later, Föllmer and Schied [8], Frittelli and Rosazza Gianin [10], and Heath [13] pointed out that the condition of positive homogeneity might be too restrictive in some applications and introduced the more general concept of a *convex risk measure* defined as a map $\rho : L^\infty \to \mathbb{R}$ satisfying properties (b), (d), and

(a') (Convexity) $\rho(\lambda X + (1 - \lambda)Y) \le \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for $\lambda \in [0, 1]$.

Clearly, any coherent risk measure is a convex risk measure.

One of the important differences between these notions and such traditional risk measures as the variance or the Value-at-Risk (V@R) is that convex and coherent risk measures are not necessarily law invariant, where *law invariance* means that $\rho(X) = \rho(Y)$ whenever X and Y have the same distribution. As a trivial example of a coherent risk measure that is not law invariant, take $\rho(X) = -\mathsf{E}_\mathsf{Q} X$, where Q is a probability measure which is absolutely continuous with respect to P and $\mathsf{Q} \neq \mathsf{P}$.

Canonical examples of coherent risk measures such as *Tail V@R* (known also as *Average V@R*, *Conditional V@R*, *Expected Shortfall*, and *Expected Tail Loss*),

$$\rho_\lambda(X) := -\inf\Big\{\mathsf{E}_\mathsf{Q} X : \mathsf{Q} \ll \mathsf{P} \text{ and } \frac{d\mathsf{Q}}{d\mathsf{P}} \leq \lambda^{-1}\Big\}, \quad \lambda \in (0,1],$$

or *Weighted V@R* (known also as the *spectral risk measure*, see e.g. [1] and [5]),

$$\rho^\mu(X) := \int_{(0,1]} \rho_\lambda(X)\mu(d\lambda), \quad \mu \text{ is a probability measure on } (0,1],$$

are law invariant. Typical examples of convex (non-coherent) risk measures such as the robust shortfall risk (see [9; Sect. 4.9]) are also law invariant. However, there exist natural derivative risk measures that are not law invariant. For example, if an agent with a random endowment X can trade in the financial market, they could be interested in the following market-adjusted convex risk measure

$$\rho'(X) := \inf_{Y \in A} \rho(X+Y),$$

where ρ is some coherent risk measure and A is the set of marketed contingent claims. Since the set A is typically not law invariant, such ρ' would not be law invariant too. Also the factor risks $\rho^f(X) := \rho(\mathsf{E}(X|Y))$ introduced by Cherny and Madan [6] are not law invariant (at the end of the introduction we give some financial interpretation of the factor risks). Other examples of such derivative measures can be found in [4; Table 1]. To sum up, basic risk measures are typically law invariant; derivative risk measures are not.

An explicit description of law invariant coherent risk measures was established by Kusuoka [16] and extended to the case of convex risk measures by Kunze [15] and independently by Frittelli and Rosazza Gianin [11] (see also [9; Th. 4.57] or [14; Th. 2.1]). Let us recall this result: *on an atomless probability space, a convex risk measure ρ is law invariant if and only if*

$$\rho(X) = \sup_{\mu \in \mathcal{M}(0,1]} (\rho^\mu(X) - \beta(\mu)), \tag{1}$$

where $\mathcal{M}(0,1]$ is the set of all probability measures on $(0,1]$ and β is a map from $\mathcal{M}(0,1]$ to $(-\infty,+\infty]$ that is not identically equal to $+\infty$. Note that, unlike the papers mentioned above, we did not assume that the risk measure ρ satisfies the Fatou property (i.e. that $\rho(X_n) \to \rho(X)$ whenever $X_n \searrow X$ a.s.) since the recent result of Jouini, Schachermayer, and Touzi [14] proves that any law invariant convex risk measure on an atomless probability space necessarily satisfies the Fatou property.

Representation (1) has a simple corollary (see [9; Cor. 4.59]): *any law invariant convex risk measure on an atomless probability space has to satisfy*

$$\rho(\mathsf{E}(X|\mathcal{G})) \leq \rho(X) \quad \textit{for any } X \in L^\infty \textit{and any } \sigma\textit{-subfield } \mathcal{G} \subseteq \mathcal{F}. \tag{2}$$

Indeed, in view of (1), to prove this corollary one just needs to notice that all ρ_λ satisfy (2), which is almost trivial. Property (2) was introduced by Leitner [17] under the name *dilatation monotonicity.* This property is economically natural since the position $\mathsf{E}(X\,|\,\mathcal{G})$ is more determined than X and thus should involve less risk.

The main result of our paper is that dilatation monotonicity implies law invariance:

Theorem 1.1. *On an atomless probability space any L^∞-continuous dilatation monotone map $R : L^\infty \to \mathbb{R}$ is law invariant.*

Using translation invariance, one can easily check that any convex risk measure satisfies $|\rho(X) - \rho(Y)| \le \|X - Y\|_{L^\infty}$. Thus, Theorem 1.1 proves the converse of the above cited corollary, and we conclude that *on an atomless probability space a convex risk measure is law invariant iff it is dilatation monotone.*

A weaker version of Theorem 1.1 was proved by Grigoriev and Leitner [12; Th. 1.11] for comonotonic additive coherent risk measures. However, the proof of Theorem 1.1 is qualitatively different from the proof of [12; Th. 1.11] and basically uses only one non-standard and surprising phenomenon: *if the probability space is atomless and $\mathsf{P}(A) = \mathsf{P}(B)$, then, for any $\varepsilon > 0$, there exists a finite sequence of σ-subfields $(\mathcal{G}_k)_{k=1}^K$ such that*

$$\left\| \mathbf{1}_B - E_K \circ \cdots \circ E_1(\mathbf{1}_A) \right\|_{L^\infty} < \varepsilon,$$

where $E_k(\cdot) = \mathsf{E}(\,\cdot\,|\,\mathcal{G}_k)$.

Remark. The statement above is true if instead of the indicator functions we have any bounded identically distributed random variables. One can easily verify this using Lemma 2.4 and the arguments from the proof of Theorem 1.1.

We prove Theorem 1.1 in the next section. In the remainder of the introduction we provide some mathematically simple but economically useful interpretations.

First, let us mention that dilatation monotonicity is closely related to *second-order stochastic dominance.* Recall that X is second-order stochastically dominated by Y ($X \preceq^2 Y$), if $\mathsf{E}u(X) \le \mathsf{E}u(Y)$ for all utility (i.e. increasing concave) functions u. To make a bridge between convex risk measures and classical expected utility preferences it is natural to study the risk measures satisfying

$$\rho(Y) \le \rho(X) \quad \text{whenever } X \preceq^2 Y.$$

We call such ρ *second-order stochastically monotone.* Clearly, any second-order stochastically monotone risk measure has to be law invariant. Conversely, by [9; Cor. 4.59] combined with [14; Th. 2.1], we conclude that any law invariant convex risk measure on an atomless probability space has to be second-order stochastically monotone.

It is well known (see e.g. [9; Cor. 2.58]) that $X \preceq^2 \mathsf{E}(X\,|\,\mathcal{G})$ for any σ-subfield $\mathcal{G}$, and thus second-order stochastic monotonicity implies dilatation monotonicity. However, dilatation monotonicity is weaker than second-order stochastic monotonicity (see e.g. [12], example on p. 42). Nevertheless, if the basis probability space is atomless, Theorem 1.1 proves that dilatation monotonicity implies law invariance, and thus all the three properties are equivalent.

Finally, let us note that dilatation monotonicity is equivalent to the property of *factor monotonicity* introduced by Cherny and Madan [6]. Let ρ be a convex risk measure, and let the random variables $Y_1, \dots, Y_M$ represent some relevant market factors (such as price of oil, the S&P 500 index, or the credit spread). The *factor risk driven by* $Y_1, \dots, Y_M$ is the functional

$$\rho^f(X; Y_1, \dots, Y_M) := \rho(\mathsf{E}(X \,|\, Y_1, \dots, Y_M)).$$

Clearly, $\rho^f(\,\cdot\,; Y_1, \dots, Y_M)$ is again a convex risk measure. From a financial point of view, factor risk represents the risk of X given the uncertainty inherent in $Y_1, \dots, Y_M$. Roughly speaking, the concept of factor risk extends that of systematic risk from the APT (to get the proper understanding, assume that $X, Y_1, \dots, Y_M$ are jointly Gaussian). In the APT, the more factors are considered, the bigger the systematic risk is. Thus, it is natural to require that

$$\rho^f(X; Y_1, \dots, Y_M) \le \rho^f(X; Y_1, \dots, Y_M, Y'_1, \dots, Y'_{M'})$$

for all $X \in L^\infty$ and all random variables $Y_1, \dots, Y_M, Y'_1, \dots, Y'_{M'}$. This property of ρ might be called *factor monotonicity*. Clearly, dilatation monotonicity implies factor monotonicity. Proposition 3.2 from [17] proves the converse.

To sum up, we get the following.

Corollary 1.2. *Suppose that the probability space is atomless. Then, for a convex risk measure ρ, the following properties are equivalent:*

(i) *law invariance;*
(ii) *dilatation monotonicity;*
(iii) *second-order stochastic monotonicity;*
(iv) *factor monotonicity;*
(v) *ρ admits a representation of the form* (1).

Each of them implies the Fatou property.

2 Proof of Theorem 1.1

Lemma 2.1. *Let $M \in \mathbb{N}$ and $A = \bigsqcup_{n=1}^{2^M} A_n$, $B = \bigsqcup_{n=1}^{2^M} B_n$ be unions of disjoint sets such that $A \cap B = \emptyset$ and $\mathsf{P}(A_n) = \mathsf{P}(B_n) = 2^{-M}\mathsf{P}(A) = 2^{-M}\mathsf{P}(B)$. Let $X \in L^\infty$ be such that $X = 1$ on A and $X = 0$ on B. Then there exist σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_K \subseteq \mathcal{F}$ such that $E_K \circ \dots \circ E_1 X = Y$, where $E_k(\cdot) = \mathsf{E}(\,\cdot\,|\mathcal{G}_k)$ and*

$$Y = \begin{cases} 2^{-M} n & \text{on } A_n,\ n = 1, \dots, 2^M, \\ 2^{-M}(n-1) & \text{on } B_n,\ n = 1, \dots, 2^M, \\ X & \text{on } (A \sqcup B)^c. \end{cases}$$

`Proof`. We proceed by induction in M.

Base. For $M = 1$, it is sufficient to consider $\mathcal{G} = \mathcal{G}(A_1 \sqcup B_2)$, where $\mathcal{G}(C)$ denotes the maximal σ-subfield of $\mathcal{F}$ containing a set C as an atom. Then $\mathsf{E}(X \,|\, \mathcal{G}) = Y$.

Step. Suppose the statement is true for M and let us prove it for $M+1$. Consider the sets $A'_n = A_{2n-1} \sqcup A_{2n}$, $B'_n = B_{2n-1} \sqcup B_{2n}$, $n = 1, \dots, 2^M$. By the induction hypothesis, there exist σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_{K'} \subseteq \mathcal{F}$ such that $E_{K'} \circ \dots \circ E_1 X = Y'$, where

$$Y' = \begin{cases} 2^{-M} n & \text{on } A'_n,\ n = 1, \dots, 2^M, \\ 2^{-M}(n-1) & \text{on } B'_n,\ n = 1, \dots, 2^M, \\ X & \text{on } (A \sqcup B)^c. \end{cases}$$

For the σ-subfields $\mathcal{G}_{K'+k} = \mathcal{G}(A_{2k-1} \sqcup B_{2k})$, $k = 1, \dots, 2^M$, it is easy to verify that $E_{K'+2^M} \circ \dots \circ E_{K'+1} \circ E_{K'} \circ \dots \circ E_1 X = Y$. □

Lemma 2.2. *Let $M \in \mathbb{N}$, $N \geq 2^M$, and $A = \bigsqcup_{n=1}^N A_n$, $B = \bigsqcup_{n=1}^N B_n$ be unions of disjoint sets such that $A \cap B = \emptyset$ and $\mathsf{P}(A_n) = \mathsf{P}(B_n) = N^{-1}\mathsf{P}(A) = N^{-1}\mathsf{P}(B)$. Let $X \in L^\infty$ satisfy*

$$X = \begin{cases} 2^{-M}n & \text{on } A_n,\ n = 1, \dots, 2^M, \\ 1 & \text{on } A_n,\ n = 2^M + 1, \dots, N, \\ 0 & \text{on } B_n,\ n = 1, \dots, N - 2^M, \\ 2^{-M}(n - N + 2^M - 1) & \text{on } B_n,\ n = N - 2^M + 1, \dots, N. \end{cases}$$

Then there exist σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_K \subseteq \mathcal{F}$ such that $E_K \circ \dots \circ E_1 X = Y$, where

$$Y = \begin{cases} 2^{-M} & \text{on } A_n,\ n = 1, \dots, N - 2^M, \\ 2^{-M}(n - N + 2^M) & \text{on } A_n,\ n = N - 2^M + 1, \dots, N, \\ 2^{-M}(n - 1) & \text{on } B_n,\ n = 1, \dots, 2^M, \\ 1 - 2^{-M} & \text{on } B_n,\ n = 2^M + 1, \dots, N, \\ X & \text{on } (A \sqcup B)^c. \end{cases}$$

Proof. We proceed by induction in N.

Base. If $N = 2^M$, then $X = Y$ and the claim is trivial.

Step. Suppose the statement is true for N and let us prove it for $N+1$. Applying the induction hypothesis to $A' = \bigsqcup_{n=1}^N A_n$ and $B' = \bigsqcup_{n=2}^{N+1} B_n$, we get σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_{K'} \subseteq \mathcal{F}$ such that $E_{K'} \circ \dots \circ E_1 X = Y'$, where

$$Y' = \begin{cases} 2^{-M} & \text{on } A_n,\ n = 1, \dots, N - 2^M, \\ 2^{-M}(n - N + 2^M) & \text{on } A_n,\ n = N - 2^M + 1, \dots, N, \\ 1 & \text{on } A_{N+1}, \\ 0 & \text{on } B_1, \\ 2^{-M}(n - 2) & \text{on } B_n,\ n = 2, \dots, 2^M + 1, \\ 1 - 2^{-M} & \text{on } B_n,\ n = 2^M + 2, \dots, N + 1, \\ X & \text{on } (A \sqcup B)^c. \end{cases}$$

For the σ-subfields $\mathcal{G}_{K'+k} = \mathcal{G}(A_{N-2^M+k+1} \sqcup B_{k+1})$, $k = 1, \dots, 2^M - 1$, it is easy to verify that $E_{K'+2^M} \circ \dots \circ E_{K'+1} \circ E_{K'} \circ \dots \circ E_1 X = Y$. □

Lemma 2.3. *Let A and B be disjoint sets such that $\mathsf{P}(A) = \mathsf{P}(B)$. Let $X \in L^\infty$ be such that $X = a$ on A and $X = b$ on B $(a, b \in \mathbb{R})$. Then, for any $\varepsilon > 0$, there exist σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_K \subseteq \mathcal{F}$ such that $\|E_K \circ \dots \circ E_1 X - Y\|_{L^\infty} < \varepsilon$, where*

$$Y = \begin{cases} b & \text{on } A, \\ a & \text{on } B, \\ X & \text{on } (A \sqcup B)^c. \end{cases}$$

Proof. Without loss of generality, $a = 1$, $b = 0$. Take $M \in \mathbb{N}$ and $N \geq 2^M$. Since $(\Omega, \mathcal{F}, \mathsf{P})$ is atomless, we can represent A as $\bigsqcup_{n=1}^N A_n$ and B as $\bigsqcup_{n=1}^N B_n$, where $\mathsf{P}(A_n) = \mathsf{P}(B_n) = N^{-1}\mathsf{P}(A) = N^{-1}\mathsf{P}(B)$. Applying Lemma 2.1 to $\bigsqcup_{n=1}^{2^M} A_n$ and

$\bigsqcup_{n=N-2^M+1}^{N} B_n$ and then Lemma 2.2 to $\bigsqcup_{n=1}^{N} A_n$ and $\bigsqcup_{n=1}^{N} B_n$, we obtain the existence of σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_{K'} \subseteq \mathcal{F}$ such that

$$Y' := E_{K'} \circ \cdots \circ E_1 X = \begin{cases} 2^{-M} & \text{on } \bigsqcup_{n=1}^{N-2^M} A_n, \\ 1 - 2^{-M} & \text{on } \bigsqcup_{n=2^M+1}^{N} B_n, \\ X & \text{on } (A \sqcup B)^c \end{cases}$$

and $Y' \in [0,1]$ on $\bigsqcup_{n=N-2^M+1}^{N} A_n \cup \bigsqcup_{n=1}^{2^M} B_n$. Taking M sufficiently large and $N > 2^{2M}$, we get $\|\mathsf{E}(Y'|\mathcal{G}(A,B)) - Y\|_{L^\infty} < \varepsilon$, where $\mathcal{G}(A,B)$ is the maximal σ-subfield of $\mathcal{F}$ containing A and B as atoms. $\square$

Lemma 2.4. *Let $X, Y \in L^\infty$ and $A \in \mathcal{F}$ be such that X and Y take on values $x_1, \dots, x_N$ on A, $\mathsf{P}(\{X = x_n\} \cap A) = \mathsf{P}(\{Y = x_n\} \cap A)$, and $X = Y$ on A^c. Then, for any $\varepsilon > 0$, there exist σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_K \subseteq \mathcal{F}$ such that $\|E_K \circ \cdots \circ E_1 X - Y\|_{L^\infty} < \varepsilon$.*

`Proof`. We proceed by induction in N.

Base. For $N = 1$ the statement is trivial since $X = Y$.

Step. Suppose the statement is true for N and let us prove it for $N+1$. First, note that it suffices to consider only the case, when $\mathsf{P}(\{X = x_1\} \cap \{Y = x_1\} \cap A) = 0$. Indeed, if we prove the lemma for this case, we can prove the general case by using $A \setminus \{X = x_1, Y = x_1\}$ instead of A.

Consider the sets $B_k = \{X = x_{k+1}, Y = x_1\} \cap A$, $k = 1, \dots, N$. We have

$$\sum_{k=1}^{N} \mathsf{P}(B_k) = \mathsf{P}(\{Y = x_1\} \cap A) = \mathsf{P}(\{X = x_1\} \cap A).$$

Since the probability space is atomless, we can find disjoint sets $C_1, \dots, C_N \subseteq \{X = x_1\} \cap A$ such that $\mathsf{P}(C_k) = \mathsf{P}(B_k)$ (note that all C_k and B_k are disjoint). Applying Lemma 2.3 to the pair (B_1, C_1), we get the existence of σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_{K_1} \subseteq \mathcal{F}$ such that

$$\|E_{K_1} \circ \cdots \circ E_1 X - Z_1\|_{L^\infty} < \frac{\varepsilon}{2N},$$

where $Z_1 := x_1 \mathbf{1}_{B_1} + x_2 \mathbf{1}_{C_1} + X \mathbf{1}_{(B_1 \cup C_1)^c}$. Applying Lemma 2.3 successively to the pairs $(B_2, C_2), \dots, (B_N, C_N)$, we get σ-subfields $(\mathcal{G}_{K_{n-1}+1}, \dots, \mathcal{G}_{K_n})_{n=2}^{N}$ such that

$$\|E_{K_n} \circ \cdots \circ E_{K_{n-1}+1} Z_{n-1} - Z_n\|_{L^\infty} < \frac{\varepsilon}{2N}, \quad n = 2, \dots, N,$$

where

$$Z_n = \begin{cases} x_1 & \text{on } \bigsqcup_{k=1}^{n} B_k, \\ x_{k+1} & \text{on } C_k,\ k = 1, \dots, n, \\ X & \text{otherwise.} \end{cases}$$

Since the linear operators E_k are contractions on L^∞, we derive that

$$\|E_{K_N} \circ \cdots \circ E_1 X - Z_N\|_{L^\infty} < \|E_{K_N} \circ \cdots \circ E_{K_1+1} Z_1 - Z_N\|_{L^\infty} + \frac{\varepsilon}{2N} < \cdots < \frac{\varepsilon}{2}.$$

The random variables Z_N and Y have the same distribution, they take on the values $(x_k)_{k=2}^{N+1}$ on the set $A \setminus \{Y = x_1\}$, and $Z_N = Y$ outside this set. Applying the induction hypothesis, we get the existence of σ-subfields $\mathcal{G}_{K_N+1}, \dots, \mathcal{G}_K$ such that $\|E_K \circ \dots \circ E_{K_N+1} Z_N - Y\|_{L^\infty} < \varepsilon/2$. By the contraction property of E_k, we get

$$\|E_K \circ \dots \circ E_1 X - Y\|_{L^\infty} < \|E_K \circ \dots \circ E_{K_N+1} Z_N - Y\|_{L^\infty} + \frac{\varepsilon}{2} < \varepsilon.$$

`Proof of Theorem 1.1.` Let $X, Y \in L^\infty$ have the same distribution. Let us approximate them by the finite step functions

$$X_n := \sum_{k=-\infty}^{\infty} \frac{k}{n} \mathbf{1}_{\{X \in \Delta_n^k\}}, \quad Y_n := \sum_{k=-\infty}^{\infty} \frac{k}{n} \mathbf{1}_{\{Y \in \Delta_n^k\}},$$

where $\Delta_n^k := \left[\frac{k}{n}, \frac{k+1}{n}\right)$. Clearly, X_n and Y_n are equal in law and $\|X - X_n\|_{L^\infty} = \|Y - Y_n\|_{L^\infty} \leq 1/n$. By Lemma 2.4, for any $\varepsilon > 0$, we can find σ-subfields $\mathcal{G}_1, \dots, \mathcal{G}_K$ such that $\|E_K \circ \dots \circ E_1 X_n - Y_n\|_{L^\infty} < \varepsilon$. Due to the dilatation monotonicity of R,

$$R(X_n) \geq R(Y_n + (E_K \circ \dots \circ E_1 X_n - Y_n)).$$

Using the continuity of R, we get $R(X_n) \geq R(Y_n)$ since ε is arbitrary. By the symmetry, $R(X_n) = R(Y_n)$. Employing the continuity of R once again, we get $R(X) = R(Y)$. □

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