BROWNIAN MOVING AVERAGES HAVE CONDITIONAL FULL SUPPORT

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Abstract. We prove that for any Brownian moving average

$$X_t = \int_{-\infty}^{t} (f(s-t) - f(s))dB_s, \quad t \geq 0,$$

the process $S = e^X$ satisfies the *conditional full support* condition introduced by Guasoni, Rásonyi, and Schachermayer [4]. This, combined with the results of [4], shows that S admits an ε-consistent price system for any $\varepsilon > 0$ and also provides the form of asymptotic superreplication prices for options of the form $g(S_T)$.

Key words: Brownian moving average, conditional full support, consistent price system, Titchmarsh convolution theorem.

1 Introduction

1. Overview. It is well known (see Soner, Shreve, and Cvitanić [12], Leventhal and Skorokhod [9], Cherny [2]) that in the Black-Scholes-Merton model with proportional transaction costs the superreplication price of a European call option converges to its trivial upper bound as the transaction cost coefficient tends to zero. The same is true for any European type contingent claim in this model (see Cvitanić, Pham, and Touzi [3]). In the recent paper [4], Guasoni, Rásonyi, and Schachermayer proved that the same result holds for a much wider class of models satisfying only a minor geometric condition termed *conditional full support* and denoted *CFS* for brevity.

This condition is as follows. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ and a continuous strictly positive $(\mathcal{F}_t)$-adapted process $(S_t)_{t\in[0,T]}$ meaning the discounted price of an asset. The CFS condition requires that, for any $t \in [0,T]$,

$$\text{supp Law}(S_u; t \leq u \leq T \mid \mathcal{F}_t) = C^+_{S_t}[t,T] \quad \text{a.s.},$$

where $C^+_x[t,T]$ is the space of continuous functions $f : [t,T] \to (0,\infty)$ with $f(t) = x$ and "supp" denotes the support (the conditional distribution here is viewed as a measure on the space $C^+[t,T]$ of continuous strictly positive functions on $[t,T]$).

Moreover, as shown in [4], under the CFS condition, for any $\varepsilon > 0$, there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ and an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale M such that $(1+\varepsilon)^{-1}S \leq M \leq (1+\varepsilon)S$.

[1]This paper would have not been written without the advise of Stanislav Molchanov, who suggested to employ the Titchmarsh convolution theorem, which is at the heart of the proof of the main theorem of the paper. I express my thanks to Martin Schweizer for having attracted my attention to the paper by Guasoni, Rásonyi, and Schachermayer. I am thankful to Miklos Rásonyi for the careful reading of the manuscript and important remarks.

The pair $(\widetilde{\mathsf{P}}, M)$ is known as an *ε-consistent price system.* Thus, CFS is a completely new geometric condition ensuring the existence of consistent price systems in models with transaction costs; in contrast, all the earlier papers (see, for example, Jouini and Kallal [6], Cherny [2]) derive the existence of consistent price systems from various sorts of the No Arbitrage condition.

Let us also mention the recent paper of Kabanov and Stricker [8] that extends the results of [4] to the multiasset framework introduced by Kabanov [7].

2. Goal of the paper. As motivated by the above discussion, the CFS condition is interesting and important. The paper [4] provides several examples of processes satisfying this condition. One of them is $S = e^X$, where X is a Fractional Brownian motion (FBM). It is well known (see Mandelbrot and Van Ness [11]) that FBM is a Brownian moving average, i.e. it can be represented as

$$X_t = \int_{-\infty}^{t} (f(s-t) - f(s))dB_s, \quad t \geq 0 \tag{1.1}$$

with a certain function $f : \mathbb{R} \to \mathbb{R}$ such that $f = 0$ on $\mathbb{R}_+$ and $\int_{-\infty}^{t}(f(s-t)-f(s))^2 ds < \infty$ for any $t \geq 0$. Let us remark that the class of moving averages includes processes that are, in a sense, more convenient for financial modelling than FBM: for example, FBM is not a semimartingale, while a moving average is a semimartingale provided that f is absolutely continuous and its derivative is square integrable on $(-\infty, 0]$ (see Cheridito [1]).

The main result of the paper is

Theorem 1.1. *Let X be a continuous process of the form* (1.1)*, where f is not zero on a set of positive Lebesgue measure. Then the process $S = e^X$ satisfies the CFS condition with respect to its natural filtration.*[2]

We also consider the CFS condition for a more general class of exponents of Gaussian processes. In the discrete time it is easy to see that the CFS condition (appropriately redefined for the discrete time case) is satisfied provided that $S = e^X$, where X is a Gaussian process such that $\mathsf{Var}(X_t - X_s \mid X_u; u \leq s) > 0$ for any $s < t$ (by Var we denote the variance). This might seem a bit surprising, but in the continuous time the corresponding result does not hold: see Example 3.1.

2 Proof of Theorem 1.1.

Let $T > 0$ and let $f \in L^2[-T, 0]$. For $g \in L^2[0, T]$, we denote by $f * g$ the convolution of f and g restricted to $[0, T]$, i.e. the function

$$(f * g)(t) = \int_0^t f(s-t)g(s)ds, \quad t \in [0, T].$$

We denote by $C_0[0, T]$ the space of continuous functions $[0, T] \to \mathbb{R}$ vanishing at zero endowed with the standard sup norm.

[2] Obviously, a process e^X satisfies the CFS condition if and only if X itself satisfies the variant of this condition for real-valued processes: for any $t \leq T$,

$$\text{supp Law}(X_u; t \leq u \leq T \mid \mathcal{F}_t) = C_{X_t}[t, T] \quad \text{a.s.},$$

where $C_x[t, T]$ is the space of continuous functions $f : [t, T] \to \mathbb{R}$ with $f(t) = x$.

Lemma 2.1. *Let $f \in L^2[-T,0]$ satisfy the condition $\int_{-\varepsilon}^0 |f(t)|dt > 0$ for any $\varepsilon > 0$. Then the space $\{f * g : g \in L^2[0,T]\}$ is dense in $C_0[0,T]$.*

Proof. If g is absolutely continuous with a square integrable derivative and $g(0) = 0$, then $(f * g)' = f * g'$. Thus, if a function $f * g$ approximates a function $h \in L^2[0,T]$ in the L^2-sense, then the function $f * G$, where $G(t) = \int_0^t g(s)ds$, approximates the function $H(t) = \int_0^t h(s)ds$ in the $C_0[0,T]$-sense. So, it is sufficient to prove that the space $\{f * g : g \in L^2[0,T]\}$ is dense in $L^2[0,T]$.

Suppose that this is not true. Then there exists a function $h \in L^2[0,T]$ not identically equal to zero such that

$$\int_0^T (f * g)(t)h(t)dt = 0 \quad \forall g \in L^2[0,T].$$

This means that

$$0 = \int_0^T \int_0^t f(s-t)g(s)h(t)dsdt = \int_0^T \int_s^T f(s-t)g(s)h(t)dtds \quad \forall g \in L^2[0,T],$$

which, in turn, is equivalent to the property

$$\int_s^T f(s-t)h(t)dt = 0, \quad \forall s \in [0,T].$$

But this is impossible due to the Titchmarsh convolution theorem (see [13] for the original proof and [5; Th. 4.3.3], [10; Lect. 16], [14; Ch. VI] for more proofs). The obtained contradiction yields the desired result. □

Proof of Theorem 1.1. Let $a \in (-\infty, 0]$ be the number such that $f = 0$ a.e. with respect to the Lebesgue measure on $[a,0]$ and $\int_{a-\varepsilon}^a |f(x)|dx > 0$ for any $\varepsilon > 0$. We can assume that $a = 0$. The case $a < 0$ is reduced to this one by considering the new Brownian motion $\widetilde{B}_t = B_{t-a} - B_{-a}$ and the new function $\widetilde{f}(x) = f(x-a)$.

We have to prove that, for any $t \in [0,T]$,

$$\operatorname{supp} \operatorname{Law}(X_u - X_t; t \le u \le T \mid \mathcal{F}_t) = C_0[t,T] \quad \text{a.s.},$$

where $C_0[t,T]$ is the space of continuous functions $f : [t,T] \to \mathbb{R}$ with $f(t) = 0$ and $\mathcal{F}_t = \sigma(X_s; s \le t)$. Obviously, it is sufficient to prove the above property with $\mathcal{F}_t$ replaced by the larger filtration $\mathcal{G}_t = \sigma(B_s : -\infty < s \le t)$. With this substitution, it is obviously sufficient to check the property only for $t = 0$. We then have

$$\begin{aligned}
&\operatorname{Law}(X_u; 0 \le u \le T \mid \mathcal{G}_0)(\omega) \\
&= \operatorname{Law}\left(\int_0^u f(v-u)dB_v + \int_{-\infty}^0 (f(v-u) - f(v))dB_v; 0 \le u \le T \,\middle|\, \mathcal{G}_0\right)(\omega) \\
&= \operatorname{Law}\left(\int_0^u f(v-u)dB_v + \varphi(u,\omega); 0 \le u \le T\right),
\end{aligned}$$

where $\varphi(\cdot,\omega)$ is the path of the process $Y = \int_{-\infty}^0 (f(v-\cdot) - f(v))dB_v$ corresponding to the elementary outcome ω.

Now, it is sufficient to prove that

$$\operatorname{supp} \operatorname{Law}\left(\int_0^u f(v-u)dB_v; u \le T\right) = C_0[0,T]. \tag{2.1}$$

It follows from the Girsanov theorem that, for any $g \in L^2[0,T]$,

$$\mathrm{Law}\biggl(\int_0^u f(v-u)dB_v; u \le T\biggr) \sim \mathrm{Law}\biggl(\int_0^u f(v-u)dB_v + \int_0^u f(v-u)g(v)dv; u \le T\biggr).$$

Hence, if a function ψ belongs to the left-hand side of (2.1), then the same is true for $\psi + \int_0^\cdot f(v-\cdot)g(v)dv$. Using now the non-emptiness of the support and recalling Lemma 2.1, we obtain (2.1), which completes the proof. □

3 Example

Let $(X_n)_{n=0,\dots,N}$ be a Gaussian random sequence such that

$$\mathsf{Var}(X_n - X_{n-1} \mid X_i; i \le n-1) > 0 \quad \forall n = 1,\dots,N. \tag{3.1}$$

Using the induction in m, it is then easy to see that X satisfies the discrete-time version of the CFS condition:

$$\operatorname{supp}\mathrm{Law}(X_i : i = n+1,\dots,m \mid X_i : i = 0,\dots,n) = \mathbb{R}^{m-n}, \quad \forall 0 \le n < m \le N. \tag{3.2}$$

Now, for the price process $S = e^X$, the discrete-time version of the Guasoni-Rásonyi-Schachermayer result (the proof is much easier in this case) now provides the existence of an ε-consistent price system for any $\varepsilon > 0$ as well as the triviality of bounds for European options on S.

Let us remark that (3.2) obviously implies (3.1), so that the latter property serves as the criterion for the CFS for discrete-time Gaussian processes.

Surprisingly enough, in continuous time such a simple criterion does not hold, as shown by the next example.

Example 3.1. Let B be a Brownian motion. For $n \in \mathbb{Z}_+$, denote $a_n = 1 - 2^{-n}$ and let

$$\begin{aligned} X_t^n = b_n &\int_0^t I(a_n \le s \le a_{n+1})dB_s \\ &+ b_n 2^{2n+3} \int_{a_n}^1 (B_{s\wedge a_{n+1}} - B_{a_n})ds \int_0^t I(s \ge a_{n+1})ds, \quad t \in [0,1]. \end{aligned}$$

The constants b_n are strictly positive and decrease to zero fast enough to ensure that

$$\sum_{n=0}^{\infty} \sup_{t\in[0,1]} |X_t^n| < \infty \quad \text{a.s.}$$

Then the process

$$X_t = \sum_{n=0}^{\infty} X_t^n, \quad t \in [0,1]$$

is continuous and Gaussian. For any $0 \le s < t \le 1$, the difference $X_t - X_s$ can be represented as $\xi_1 + \xi_2$, where ξ_1 is $\sigma(X_u; u \le s)$-measurable and ξ_2 is non-degenerate and depends on the increments of B after time s. Hence,

$$\mathsf{Var}(X_t - X_s \mid X_u; u \le s) > 0 \quad \forall 0 \le s < t \le 1.$$

But on the other hand,

$$\int_0^1 X_t dt = \sum_{n=0}^{\infty} \int_0^1 X_t^n dt = \sum_{n=0}^{\infty} b_n \int_{a_n}^1 (B_{s\wedge a_{n+1}} - B_{a_n})ds \biggl[1 + 2^{2n+3} \int_{a_{n+1}}^1 (s - a_{n+1})ds\biggr] = 0,$$

so that the CFS condition is violated for X already for $t = 0$.

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