CONVERGENCE OF SOME INTEGRALS ASSOCIATED WITH BESSEL PROCESSES

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Abstract. We study the convergence of the Lebesgue integrals for the processes $f(\rho_t)$. Here, $(\rho_t, t \geq 0)$ is the δ-dimensional Bessel process started at $\rho_0 \geq 0$ and f is a positive Borel function. The obtained results are applied to prove that two Bessel processes of different dimensions have singular distributions.

Key words and phrases. Bessel processes, the Engelbert-Schmidt Zero-One law, Brownian local time, regular continuous strong Markov processes, singularity of distributions.

1 Introduction

1. Let us first introduce some notions. Consider the stochastic differential equation

$$Z_t = z_0 + \delta t + 2 \int_0^t \sqrt{|Z_s|}\, dB_s, \tag{1.1}$$

where $z_0 \geq 0$, $\delta \geq 0$ and $(B_t, t \geq 0)$ is the standard linear Brownian motion. This equation is known to have the unique positive strong solution (see [14, ch. IX, Theorem 3.5]). The process $\rho_t = \sqrt{Z_t}$ is called the δ-dimensional Bessel process started at $\rho_0 = \sqrt{z_0}$. For more information about Bessel processes, see [3], [12] and [14].

If $\delta > 1$, the δ-dimensional Bessel process satisfies the following stochastic differential equation:

$$\rho_t = \rho_0 + \int_0^t \frac{\delta - 1}{2\rho_s}\, ds + B_t. \tag{1.2}$$

The δ-dimensional Bessel process is a *regular continuous strong Markov process* with the *scale function*

$$s(x) = \begin{cases} -x^{2-\delta} & \text{if } \delta > 2, \\ 2 \ln x & \text{if } \delta = 2, \\ x^{2-\delta} & \text{if } 0 \leq \delta < 2. \end{cases} \tag{1.3}$$

The *speed measure* of the δ-dimensional Bessel process has the following density with respect to the Lebesgue measure

$$\begin{cases} (\delta/2-1)^{-1}x^{\delta-1} & \text{if } \delta > 2, \\ x & \text{if } \delta = 2, \\ (1-\delta/2)^{-1}x^{\delta-1} & \text{if } 0 \le \delta < 2. \end{cases} \tag{1.4}$$

For the definition of the regular Markov process, the scale function and the speed measure, see [14, ch. VII, section 3]. The formulas (1.3) and (1.4) can be found in [14, ch. XI, section 1].

It follows from the explicit form of the scale function that

$$\delta \ge 2 \implies \mathsf{P}\{\forall t > 0,\ \rho_t > 0\} = 1, \tag{1.5}$$

$$0 \le \delta < 2 \implies \mathsf{P}\{\exists t \ge 0 : \rho_t = 0\} = 1. \tag{1.6}$$

The point $x = \{0\}$ is an entrance boundary if $\delta \ge 2$, a reflecting boundary if $0 < \delta < 2$, and an absorbing point if $\delta = 0$.

The Bessel process is recurrent for $0 < \delta \le 2$ and is transient for $\delta > 2$:

$$\delta > 2 \implies \lim_{t\to\infty} \rho_t = \infty \quad \text{a.s.}, \tag{1.7}$$

$$\delta = 2 \implies \forall x > 0,\ \mathsf{P}\{\exists t \ge 0 : \rho_t = x\} = 1, \tag{1.8}$$

$$0 < \delta < 2 \implies \forall x \ge 0,\ \mathsf{P}\{\exists t \ge 0 : \rho_t = x\} = 1. \tag{1.9}$$

2. We will now recall some properties of the *Brownian local times.* The proofs can be found, for example, in [14, ch. VI, section 1]. Let B_t be the Brownian motion started at $B_0 \in \mathbb{R}$. The local time of B_t at a point $x \in \mathbb{R}$ is the process L_t^x such that

$$(B_t - x)^- = (B_0 - x)^- - \int_0^t I(B_s \le x)\, dB_s + \frac{1}{2}L_t^x, \tag{1.10}$$

where $z^- = -(z \wedge 0)$. Equality (1.10) is called *Tanaka's formula.*

There exists a bicontinuous modification of the process $(L_t^x,\ x \in \mathbb{R}, t \ge 0)$. In what follows, we will always deal with this modification.

For any positive Borel function h and any stopping time T, we have

$$\int_0^T h(B_s)\, ds \overset{\text{a.s.}}{=} \int_{\mathbb{R}} h(x)\, L_T^x\, dx. \tag{1.11}$$

For any $a \in \mathbb{R}$, the measure dL_t^a is a.s. carried by the set $\{t : B_t = a\}$. This, in view of the continuity of L_t^x in x, implies that, for $a \le B_0 \le b$,

$$\mathsf{P}\Big\{\forall x \notin [a,b],\ L^x_{T_a(B)\wedge T_b(B)} = 0\Big\} = 1, \tag{1.12}$$

where $T_c(B) = \inf\{s \ge 0 :\ B_s = c\}$ $(c \in \mathbb{R})$.

The local time of the Brownian motion started at $B_0 \in \mathbb{R}$ has the following property:

$$\mathsf{P}\Big\{\forall \varepsilon > 0,\ L_\varepsilon^{B_0} > 0\Big\} = 1. \tag{1.13}$$

3. Before proceeding to the results of this paper, we will formulate the *Engelbert-Schmidt Zero-One law.* For the proof, see [5] (and also [11, ch. 3, Proposition 6.27]).

Proposition 1.1. *Let $(B_t,\, t \geq 0)$ be the Brownian motion and f be a positive Borel function. The following assertions are equivalent:*

a) $\mathsf{P}\big\{\forall t > 0,\ \int_0^t f(B_s)\, ds < \infty\big\} > 0$;

b) $\mathsf{P}\big\{\forall t > 0,\ \int_0^t f(B_s)\, ds < \infty\big\} = 1$;

c) f *is locally integrable.*

In this paper, we prove the similar statement for Bessel processes (Corollary 2.3). We present a proof of even a stronger assertion (Theorem 2.2). This theorem states that there exists a stopping time $\xi \leq \infty$ such that

$$\mathsf{P}\left\{\forall t < \xi,\ \int_0^t f(\rho_s)\, ds < \infty\right\} = 1, \tag{1.14}$$

$$\mathsf{P}\left\{\exists t > \xi :\ \int_0^t f(\rho_s)\, ds < \infty\right\} = 0. \tag{1.15}$$

Moreover, $\xi = T_l(\rho) \wedge T_r(\rho)$. Here, l and r satisfy the inequalities $-\infty \leq l \leq \rho_0 \leq r \leq +\infty$ and are easily determined by the function f.

It is not clear from (1.14) and (1.15) whether the integral

$$\int_0^\xi f(\rho_s)\, ds. \tag{1.16}$$

converges or not. The answer to this problem is given by Theorem 2.5. The convergence of the integral

$$\int_0^\infty f(\rho_s)\, ds. \tag{1.17}$$

is treated in Theorems 2.6 and 2.7.

Some of the obtained results are already known. The convergence of the Lebesgue integrals for the functions $f(\rho_t)$ is studied in [8]. The authors of that paper prove the Zero–One law for the case $\delta \geq 2$, $\rho_0 > 0$. The case $\delta \geq 2$, $\rho_0 = 0$ was considered in [13], [16]. The case $0 < \delta < 2$ has not previously been investigated.

The convergence of (1.17) was studied in [8], [16] for the case $\delta \geq 2$.

The convergence of integral (1.16) has not previously been considered.

The similar problems were investigated for the processes other than the Bessel processes. Thus, the papers [7], [16] deal with (some) local martingales and the paper [9] treats the Brownian motion with a drift.

The main results of this paper are formulated in Section 2. The proofs are given in Section 3. In Section 4, the obtained results are applied to prove that two Bessel processes of different dimensions have singular distributions. It follows from [15] that the distributions of two Bessel processes of dimensions $\eta > \delta > 2$ started at zero are singular on the σ-field $\mathcal{F}_{0+} = \bigcap_{\varepsilon>0} \sigma(X_s,\, s \leq \varepsilon)$. We prove this assertion with another method employing no restriction $\delta > 2$. We also prove the singularity of distributions on the σ-field $\mathcal{F}_\infty = \sigma(X_s,\, s \geq 0)$ for two Bessel processes of different dimensions, started at $\rho_0 > 0$.

2 Main Results

This section contains the main theorems and the corollaries.

Definition 2.1. A positive Borel function g on $[0,\infty)$ is locally integrable at a point $a \geq 0$ if there exists $\varepsilon > 0$ such that

$$\int_{(a-\varepsilon)\vee 0}^{a+\varepsilon} g(x)\,dx < \infty.$$

The set of all points at which g is integrable is denoted by C_g. This is an open subset of $[0,\infty)$.

We will now proceed to one of the main results of this paper.

Theorem 2.2. *Let ρ_t be the δ-dimensional Bessel process ($\delta \geq 0$) started at $\rho_0 \geq 0$ (if $\delta = 0$, then $\rho_0 > 0$). Let f be a positive Borel function on $[0,\infty)$. Set*

$$g(x) = \begin{cases} x\,f(x) & \text{if } \delta > 2, \\ x\,(|\ln x| \vee 1)\,f(x) & \text{if } \delta = 2, \\ x^{\delta-1}\,f(x) & \text{if } 0 < \delta < 2, \\ x\,f(x) & \text{if } \delta = 0. \end{cases}$$

Then, for the stopping time $\xi = \inf\{s \geq 0 : \rho_s \notin C_g\}$, we have

$$\mathsf{P}\left\{\forall t < \xi,\ \int_0^t f(\rho_s)\,ds < \infty\right\} = 1,$$
$$\mathsf{P}\left\{\exists t > \xi : \int_0^t f(\rho_s)\,ds < \infty\right\} = 0.$$

`Remarks`. (i) The paper [6, Lemma 1] contains the similar statement for the Brownian motion.

(ii) It is obvious that

$$\xi = T_l(\rho) \wedge T_r(\rho), \tag{2.1}$$

where

$$r = \sup\{x \geq \rho_0 : x \in C_g\}, \tag{2.2}$$
$$l = \inf\{x \leq \rho_0 : x \in C_g \cup (-\infty, 0)\}. \tag{2.3}$$

In particular, if $C_g \supseteq [0, \rho_0]$, then $l = -\infty$, and therefore, $T_l(\rho) = \infty$ a.s. (since the process ρ_t is positive). Here, l can be assigned any strictly negative value but not zero since it may happen that $\mathsf{P}\{T_0(\rho) < \infty\} > 0$. □

Corollary 2.3. (i) *Suppose that $\delta \geq 2$ and $\rho_0 > 0$. The following assertions are equivalent:*

a) $\mathsf{P}\{\forall t > 0,\ \int_0^t f(\rho_s)\,ds < \infty\} = 1$;

b) *f is locally integrable on $(0,\infty)$, i.e. $C_f \supseteq (0,\infty)$.*

(ii) *Let $0 < \delta < 2$ or $\rho_0 = 0$. The following assertions are equivalent:*

a) $\mathsf{P}\{\forall t > 0,\ \int_0^t f(\rho_s)\,ds < \infty\} > 0$;

b) $\mathsf{P}\{\forall t > 0,\ \int_0^t f(\rho_s)\,ds < \infty\} = 1$;

c) *g is locally integrable on $[0,\infty)$, i.e. $C_g = [0,\infty)$.*

Proof. **(i)** In view of (1.5), the conditions $\delta \geq 2$, $\rho_0 > 0$ imply that $\mathsf{P}\{\forall t \geq 0,\ \rho_t > 0\} = 1$. Then it follows from Theorem 2.2 that the assertion a) is equivalent to the local integrability of g on $(0,\infty)$. This, in turn, is equivalent to the local integrability of f on $(0,\infty)$.

(ii) From (1.7)–(1.9) and the conditions $0 < \delta < 2$ or $\rho_0 = 0$, we get that $\forall x \geq 0$, $\mathsf{P}\{\exists t \geq 0 : \rho_t = x\} = 1$. Applying Theorem 2.2, one completes the proof. □

Corollary 2.4. *Suppose that $\delta > 0$, $\rho_0 = 0$ and $d \in \mathbb{R}$. Then*

$$d > -(2 \wedge \delta) \Longrightarrow \mathsf{P}\left\{\forall t > 0,\ \int_0^t \rho_s^d\, ds < \infty\right\} = 1,$$

$$d \leq -(2 \wedge \delta) \Longrightarrow \mathsf{P}\left\{\forall \varepsilon > 0,\ \int_0^\varepsilon \rho_s^d\, ds = \infty\right\} = 1.$$

It is not clear from Theorem 2.2 whether integral (1.16) converges or not. Taking (1.14) and (2.1) into account, we see that, in order to investigate (1.16), one should study the convergence of integrals

$$\int_{T_l(\rho)-\varepsilon}^{T_l(\rho)} f(\rho_s)\, ds, \quad \varepsilon > 0$$

on the set $\{T_l(\rho) < \infty\}$ and the convergence of integrals

$$\int_{T_r(\rho)-\varepsilon}^{T_r(\rho)} f(\rho_s)\, ds, \quad \varepsilon > 0$$

on the set $\{T_r(\rho) < \infty\}$. Such integrals are considered in the following theorem.

Theorem 2.5. *Suppose that $\delta \geq 0$, $\rho_0 \geq 0$, $\gamma \geq 0$ and $\gamma \neq \rho_0$. Set*

$$g(x) = |x-\gamma|\, I(x \in [\gamma, \rho_0])\, f(x).$$

Here, $[\gamma, \rho_0]$ denotes the closed interval with the endpoints γ and ρ_0 (so that γ may be greater than ρ_0). If $\gamma \in C_g$, then

$$\mathsf{P}\left(\{T_\gamma < \infty\} \cap \left\{\exists \varepsilon > 0 : \int_{T_\gamma(\rho)-\varepsilon}^{T_\gamma(\rho)} f(\rho_s)\, ds < \infty\right\}\right) = \mathsf{P}\{T_\gamma < \infty\}. \tag{2.4}$$

If $\gamma \notin C_g$, then

$$\mathsf{P}\left(\{T_\gamma < \infty\} \cap \left\{\forall \varepsilon > 0 : \int_{T_\gamma(\rho)-\varepsilon}^{T_\gamma(\rho)} f(\rho_s)\, ds = \infty\right\}\right) = \mathsf{P}\{T_\gamma < \infty\}. \tag{2.5}$$

We can apply Theorem 2.5 to study the convergence of (1.16) on the set $\{\xi < \infty\}$. The following two theorems are related to the convergence of (1.17).

Theorem 2.6. (i) *Suppose that* $\delta > 2$ *and* $\rho_0 > 0$. *Then*

$$\int_{\rho_0}^{\infty} x\, f(x)\, dx < \infty \implies \mathsf{P}\Big(\{\xi = \infty\} \cap \Big\{\int_0^{\infty} f(\rho_s)\, ds < \infty\Big\}\Big) = \mathsf{P}\{\xi = \infty\}, \quad (2.6)$$

$$\int_{\rho_0}^{\infty} x\, f(x)\, dx = \infty \implies \int_0^{\infty} f(\rho_s)\, ds = \infty \quad a.s. \quad (2.7)$$

(ii) *If* $\delta > 2$ *and* $\rho_0 = 0$, *then*

$$\int_0^{\infty} x\, f(x)\, dx < \infty \implies \int_0^{\infty} f(\rho_s)\, ds < \infty \quad a.s., \quad (2.8)$$

$$\int_0^{\infty} x\, f(x)\, dx = \infty \implies \int_0^{\infty} f(\rho_s)\, ds = \infty \quad a.s. \quad (2.9)$$

Remark. The situation may occur where $\mathsf{P}\{\xi < \infty\} > 0$ and $\mathsf{P}\{\xi = \infty\} > 0$. This is possible if $\delta > 2$, $\rho_0 > 0$ and f is locally integrable on $[\rho_0, \infty)$ but is not locally integrable on $(0, \rho_0)$. In that event, integral (1.17) diverges a.s. on the set $\{\xi < \infty\}$. Its convergence on the set $\{\xi = \infty\}$ depends on the convergence of the integral $\int_{\rho_0 - 0}^{\infty} x\, f(x)\, dx$. □

The situation is completely different if $0 < \delta \le 2$.

Theorem 2.7. *If* $0 < \delta \le 2$ *and* $\rho_0 \ge 0$, *then*

$$\int_0^{\infty} f(x)\, dx > 0 \implies \int_0^{\infty} f(\rho_s)\, ds = \infty \quad a.s.$$

The following statement is a consequence of Theorem 2.6.

Corollary 2.8. *Suppose that* $\delta > 2$, $\rho_0 > 0$ *and* $d \in \mathbb{R}$. *Then*

$$d < -2 \implies \int_0^{\infty} \rho_s^d\, ds < \infty \quad a.s.,$$

$$d \ge -2 \implies \int_0^{\infty} \rho_s^d\, ds = \infty \quad a.s.$$

3 The Proofs

We will first prove a few lemmas.

Lemma 3.1. *Suppose that* $\delta \ne 2$ *and* $(B_t,\, t \ge 0)$ *is the Brownian motion started at* $x_0 \ge 0$. *Set*

$$\lambda = (2 - \delta)^{-1}, \qquad \sigma(x) = \lambda^2\, x^{\lambda(2\delta - 2)}\, I(x > 0),$$

$$A_t = \int_0^t \sigma(B_s)\, ds, \qquad \tau_t = \inf\{s \ge 0 : A_s > t\}.$$

(i) *Suppose that* $\delta > 2$ *and* $x_0 > 0$ *or* $0 < \delta < 2$ *and* $x_0 \ge 0$. *Then the process* $(B_{\tau_t},\, t \ge 0)$ *has the same law as* $(\rho_t^{2-\delta},\, t \ge 0)$. *Here,* ρ_t *is the* δ*-dimensional Bessel process started at* $\rho_0 = x_0^{\lambda}$.

(ii) *Suppose that* $\delta = 0$. *Then the process* $(B_{\tau_t},\, t \le T_0(B))$ *has the same law as* $(\rho_t^2,\, t \le T_0(\rho))$. *Here,* ρ_t *is the zero-dimensional Bessel process started at* $\rho_0 = \sqrt{x_0}$.

Lemma 3.2. *Let $(B_t,\, t \geq 0)$ be the Brownian motion started at $x_0 \in \mathbb{R}$. Set*

$$\sigma(x) = 4^{-1} \exp x, \qquad A_t = \int_0^t \sigma(B_s)\, ds, \qquad \tau_t = \inf\{s \geq 0 : A_s > t\}.$$

Then the process $(B_{\tau_t},\ t \geq 0)$ has the same law as $(2 \ln \rho_t,\ t \geq 0)$. Here, ρ_t is the 2-dimensional Bessel process started at $\rho_0 = \exp(x_0/2)$.

Remark. In what follows, we will always use the notation λ for $(2-\delta)^{-1}$. □

Proof of Lemma 3.1. We will give the proof only for the case $\delta > 2$. Let a and b satisfy the condition $0 < a < x_0 < b < \infty$. Set $X_t = B_{\tau_t}$, $X_t^{ab} = X_{t \wedge T_a(X) \wedge T_b(X)}$. The process X_t^{ab} is a regular continuous strong Markov process on $[a, b]$ with the scale function $s_1(x) = x$. Its speed measure $m_1(dx)$ is given by the density $2\sigma(x)$ (see [14, ch. X, Theorem 2.18]).

The process ρ_t is a regular continuous strong Markov process on $(0, \infty)$ with the scale function $s_2(x) = -x^{2-\delta}$. Its speed measure $m_2(dx)$ has the density $-2\lambda\, x^{\delta-1}$ (see (1.3), (1.4)). The function $F : x \mapsto x^{2-\delta}$ is a homeomorphism of $(0, \infty)$ onto itself. The process $Y_t = F(\rho_t)$ is a regular continuous strong Markov process on $(0, \infty)$ with the scale function $s_3(x) = -s_2(F^{-1}(x))$ and the speed measure $m_3 = m_2 \circ F^{-1}$. It is easy to verify that $s_3(x) = x$ and m_3 has the density $2\sigma(x)$.

Thus, the processes X_t^{ab} and $Y_t^{ab} = Y_{t \wedge T_a(Y) \wedge T_b(Y)}$ have the same speed measure and the same scale function. Therefore, they have the same law (see [4, ch. 15, section 3]). The process Y_t (and hence, X_t) reaches neither 0 nor ∞ (see (1.5)). Since a and b are arbitrary, we deduce that $(X_t,\ t \geq 0)$ and $(Y_t,\ t \geq 0)$ have the same law. □

Lemma 3.1 for the cases $0 < \delta < 2$ and $\delta = 0$ as well as Lemma 3.2 are proved in the same way.

Lemma 3.3. *Let $(B_t,\, t \geq 0)$ be the Brownian motion started at $a \in \mathbb{R}$ and let $b < a$. Set*

$$Z_\theta^{ab} = L_{T_b(B)}^{\theta+b}, \quad \theta \geq 0.$$

Here, L_t^x is the local time of B_t at a point x. Then Z_θ^{ab} has the following properties:

(i) *the process $(Z_\theta^{ab},\ 0 \leq \theta \leq a-b)$ has the same distribution as $(|W_\theta|^2,\ 0 \leq \theta \leq a-b)$, where W_θ is the two-dimensional Brownian motion started at zero;*

(ii) *for any $\theta \geq 0$,*

$$\mathsf{E} Z_\theta^{ab} \leq 2(a-b) \wedge 2\theta.$$

Proof. **(i)** This statement is a straightforward consequence of the Ray-Knight theorem (see [14, ch. XI, Theorem 2.2]) and the scaling property of the Brownian motion.

(ii) Let us fix $\theta > a - b$. It follows from (1.10) that the process $M_t = (B_t - (\theta + b))^- - \frac{1}{2} L_t^{\theta+b}$ is a local martingale. Hence, $M_{t \wedge T_b(B)}$ is a local martingale bounded above. Therefore,

$$\begin{aligned} \mathsf{E}\left[\theta - \frac{1}{2} Z_\theta^{ab}\right] &= \mathsf{E}\left[\theta - \frac{1}{2} L_{T_b(B)}^{\theta+b}\right] = \mathsf{E} M_{T_b(B)} \geq \mathsf{E} M_0 \\ &= (a - (\theta + b))^- = -a + \theta + b. \end{aligned}$$

Thus, $\mathsf{E} Z^{ab}_\theta \le 2(a-b)$ for any $\theta > a-b$. This, combined with (i), completes the proof. □

`Remark`. The equality is valid in statement (ii) of Lemma 3.3. This follows from [1, formula 1.2.3.1]. □

We will further need the following statement proved in [2].

Proposition 3.4. *Suppose that $-\infty \le p < q \le +\infty$ and μ is a measure (positive but not necessarily finite) on (p,q). Let $(U_t,\, p<t<q)$ be a random process with measurable sample paths on (p,q) such that $\mathsf{E}|U_t| < \infty$ for any $t \in (p,q)$. Suppose that there exist constants $d>1$, $c>0$ for which*

$$\mathsf{E}|U_t|^d \le c\,(\mathsf{E}|U_t|)^d, \quad p<t<q.$$

Then

$$\int_p^q |U_t|\,\mu(dt) < \infty \ \ a.s. \iff \int_p^q \mathsf{E}|U_t|\,\mu(dt) < \infty.$$

`Proof of Theorem 2.2`. We will give the proof only for $\delta > 2$. Let us first consider the case $\rho_0 > 0$. Thanks to Lemma 3.1 (i), we may write $\rho_t = B^\lambda_{\tau_t}$, $t \ge 0$.

Let $\rho_0 \in C_g$. Then $\xi = T_l(\rho) \wedge T_r(\rho)$, where l and r are defined in (2.2) and (2.3). Note that $l < \rho_0 < r$. Let $\alpha,\ \beta \in (0,\infty)$ be arbitrary constants satisfying the inequality $l < \alpha < \rho_0 < \beta < r$. Set

$$a = \alpha^{2-\delta}, \quad b = \beta^{2-\delta},$$
$$T_1 = T_\alpha(\rho) \wedge T_\beta(\rho), \quad T_2 = T_a(B) \wedge T_b(B).$$

Using the change of variables formula and taking (1.11) into account, one gets

$$\int_0^{T_1} f(\rho_s)\,ds = \int_0^{T_1} f(B^\lambda_{\tau_s})\,ds = \int_0^{T_2} f(B^\lambda_s)\,dA_s \tag{3.1}$$
$$= \int_0^{T_2} f(B^\lambda_s)\,\sigma(B_s)\,ds \overset{\text{a.s.}}{=} \int_0^\infty h(x)\,\sigma(x)\,L^x_{T_2} dx. \tag{3.2}$$

Here, $h(x) = f(x^\lambda)$ and $\sigma(x)$ is defined in Lemma 3.1 (i). It follows from the choice of α and β that σ is bounded on $[a,b]$ and h is integrable on $[a,b]$. The local time $L^x_{T_2}$ is continuous in x. Thus, equality (3.1), combined with (1.12), implies that

$$\mathsf{P}\left\{\int_0^{T_1} f(\rho_s)\,ds < \infty\right\} = 1.$$

Since α and β are arbitrary, we conclude that

$$\mathsf{P}\left\{\forall t < T_l(\rho) \wedge T_r(\rho),\ \int_0^t f(\rho_s)\,ds < \infty\right\} = 1.$$

Suppose that $\rho_0 > 0$, $\rho_0 \notin C_g$. Then $\xi = 0$. Let $\alpha,\ \beta$ be arbitrary constants such that $0 < \alpha < \rho_0 < \beta < \infty$. Equality (3.1) remains valid with the same notations as above. It follows from (1.13) that $L^{x_0}_{T_2} > 0$ a.s. The conditions $\rho_0 > 0$ and $\rho_0 \notin C_g$ imply that $x_0 \notin C_h$. From (3.1), we obtain

$$\mathsf{P}\left\{\int_0^{T_1} f(\rho_s)\,ds = \infty\right\} = 1.$$

As α and β were chosen arbitrarily, we deduce that

$$\rho_0 > 0,\ \rho_0 \notin C_g \implies \mathsf{P}\left\{\forall\varepsilon > 0,\ \int_0^{\varepsilon} f(\rho_s)\,ds = \infty\right\} = 1. \tag{3.3}$$

This conclusion, in view of the strong Markov property of ρ_t, implies that, for $\rho_0 > 0$, $\rho_0 \in C_g$,

$$\mathsf{P}\left\{\exists\varepsilon > 0 : \int_{\xi}^{\xi+\varepsilon} f(\rho_s)\,ds < \infty\right\} = 0.$$

Thus, for $\rho_0 > 0$, the proof is completed.

We will now consider the case $\rho_0 = 0$. Suppose first that $\rho_0 \in C_g$. Then $\xi = T_r(\rho)$. Let α and β satisfy the inequality $0 < \alpha < \beta < r$. The process

$$\widetilde{\rho}_t = \rho_{t+T_\alpha(\rho)} - \rho_{T_\alpha(\rho)}, \quad t \geq 0$$

is the δ-dimensional Bessel process started at α. By Lemma 3.1 (i), we can write $\widetilde{\rho}_t = B^{\lambda}_{\tau_t}$, where B_t is the Brownian motion started at $a = \alpha^{2-\delta}$. Set $b = \beta^{2-\delta}$ and note that $0 < b < a$. We have

$$\begin{aligned}\int_{T_\alpha(\rho)}^{T_\beta(\rho)} f(\rho_s)\,ds &= \int_0^{T_\beta(\widetilde{\rho})} f(\widetilde{\rho}_s)\,ds = \int_0^{T_b(B)} f(B^{\lambda}_s)\,\sigma(B_s)\,ds \\ &\overset{\text{a.s.}}{=} \int_b^{\infty} f(x^{\lambda})\,\sigma(x)\,L^x_{T_b(B)}\,dx = \int_0^{\infty} h(x)\,\sigma(x+b)\,Z^{ab}_x\,dx. \end{aligned} \tag{3.4}$$

Here,

$$h(x) = f\left((x+b)^{\lambda}\right), \quad x > 0, \qquad Z^{ab}_x = L^{x+b}_{T_b(B)}, \quad x > 0.$$

It is easy to verify the equalities

$$\begin{aligned}\int_0^{\infty} h(y)\,(y+b)\,\sigma(y+b)\,dy &= \lambda^2 \int_0^{\infty} h(y)\,(y+b)^{\lambda\delta}\,dy \\ &= \lambda^2 \int_b^{\infty} f(y^{\lambda})\,y^{\lambda\delta}\,dy = |\lambda| \int_0^{\beta} x\,f(x)\,dx < \infty. \end{aligned} \tag{3.5}$$

Taking Lemma 3.3 (ii) into account, we arrive at

$$\mathsf{E}\left[\int_0^{\infty} h(x)\,\sigma(x+b)\,Z^{ab}_x\,dx\right] \leq 2\int_0^{\infty} h(x)\,\sigma(x+b)\,x\,dx = \Theta < \infty,$$

where Θ depends only on β. This inequality, together with (3.4), implies that

$$\mathsf{E}\left[\int_{T_\alpha(\rho)}^{T_\beta(\rho)} f(\rho_s)\,ds\right] \leq \Theta = \Theta(\beta).$$

Since α is arbitrary, we get

$$\mathsf{E}\left[\int_0^{T_\beta(\rho)} f(\rho_s)\,ds\right] \le \Theta < \infty.$$

Due to the choice of β, we deduce that

$$\mathsf{P}\left\{\forall t < T_r(\rho),\ \int_0^t f(\rho_s)\,ds < \infty\right\} = 1.$$

On the other hand, the strong Markov property for ρ_t, together with (3.3), implies that

$$\mathsf{P}\left\{\exists t > T_r(\rho) : \int_0^t f(\rho_s)\,ds < \infty\right\} = 0.$$

Now, let $\rho_0 = 0$ and $\rho_0 \notin C_g$. Then $\xi = 0$. We must show that

$$\mathsf{P}\left\{\forall \varepsilon > 0,\ \int_0^\varepsilon f(\rho_s)\,ds = \infty\right\} = 1. \tag{3.6}$$

If zero is a limit point for the set $(0,\infty) \setminus C_g$, then (3.6) immediately follows from (3.3) and the strong Markov property of ρ_t. Because of this, we will assume that there exists $\gamma > 0$ such that $(0,\gamma) \in C_g$. Let α and β satisfy the inequality $0 < \alpha < \beta < \gamma$. The following equalities are derived similarly to (3.5):

$$\int_0^z (x+b)\,h(x)\,\sigma(x+b)\,dx = \int_{(b+z)^\lambda}^\beta x\,f(x)\,dx < \infty, \tag{3.7}$$

$$\int_0^\infty (x+b)\,h(x)\,\sigma(x+b)\,dx = \int_0^\beta x\,f(x)\,dx = \infty. \tag{3.8}$$

We use here the same notations as in (3.5). It follows from (3.8) that

$$\int_0^\infty x\,h(x)\,\sigma(x+b)\,dx = \infty. \tag{3.9}$$

Let $(W_t,\, t \ge 0)$ be the two-dimensional Brownian motion started at zero. Set

$$D = \left\{\omega : \int_0^\infty h(x)\,\sigma(x+b)\,|W_x(\omega)|^2\,dx = \infty\right\}.$$

Inequality (3.7) implies that D belongs to the tail σ-field $\mathcal{G} = \bigcap_{t>0} \sigma(W_s,\ s > t)$. Blumenthal's Zero-One law, combined with the time-inversion property of the Brownian motion, implies that $\mathsf{P}(D)$ equals either 0 or 1. Due to Proposition 3.4 and equality (3.9), we have $\mathsf{P}(D) > 0$. Therefore,

$$\int_0^\infty h(x)\,\sigma(x+b)\,|W_x|^2\,dx = \infty \quad \text{a.s.}$$

Taking Lemma 3.3 (i) into account, we deduce that

$$\int_0^{a-b} h(x)\,\sigma(x+b)\,Z_x^{ab}\,dx \xrightarrow[a\to\infty]{\mathsf{P}} \infty.$$

Combining this with (3.4), we obtain

$$\int_{T_\alpha(\rho)}^{T_\beta(\rho)} f(\rho_s)\, ds \xrightarrow[\alpha\to 0]{\mathsf{P}} \infty.$$

On the other hand,

$$\int_{T_\alpha(\rho)}^{T_\beta(\rho)} f(\rho_s)\, ds \xrightarrow[\alpha\to 0]{\text{a.s.}} \int_0^{T_\beta(\rho)} f(\rho_s)\, ds.$$

Here, β can be arbitrarily small. Thus, (3.6) is verified and the proof of Theorem 2.2 is completed. □

The proof of Theorem 2.5 is based on the following statement (see [14, ch. VII, Corollary 4.6]).

Proposition 3.5. *Let $(B_t,\, t \ge 0)$ be the Brownian motion started at $a \in \mathbb{R}$ and let $c < a$. Let $(V_t,\, t \ge 0)$ be the three-dimensional Bessel process started at zero. Set $S_{a-c}(V) = \sup\{s \ge 0 : V_s = a - c\}$. Then the process $(B_t,\ 0 \le t \le T_c(B))$ has the same distribution as*

$$V_{S_{a-c}(V)-t} + c, \quad 0 \le t \le S_{a-c}(V).$$

`Proof of Theorem 2.5.` We will give the proof for $\delta > 2$. In view of (1.5), we may consider only $\gamma > 0$. Suppose that $0 < \rho_0 < \gamma$. In this case, $\mathsf{P}\{T_\gamma(\rho) < \infty\} = 1$. We may write $\rho_t = B^\lambda_{\tau_t}$ as above. Let α be an arbitrary constant satisfying the inequality $\rho_0 < \alpha < \gamma$. Set $a = \alpha^{2-\delta}$, $c = \gamma^{2-\delta}$. Thanks to Proposition 3.5, we have

$$\begin{aligned}\int_{T_\alpha(\rho)}^{T_\gamma(\rho)} f(\rho_s)\, ds &= \int_{T_a(B)}^{T_c(B)} f(B_s^\lambda)\, \sigma(B_s)\, ds \\ &\overset{\text{law}}{=} \int_0^{S_{a-c}(V)} f((c+V_s)^\lambda)\, \sigma(c+V_s)\, ds.\end{aligned}$$

Here, V_t is the three-dimensional Bessel process started at zero. Keeping in mind that $S_\varepsilon \xrightarrow[\varepsilon\to 0]{\text{a.s.}} 0$, $c > 0$ and employing Theorem 2.2, we obtain the following implications:

$$\begin{aligned}
&\exists \varepsilon > 0 : \int_{\gamma-\varepsilon}^{\gamma} (\gamma - x)\, f(x)\, dx < \infty \implies \\
&\exists \varepsilon > 0 : \int_{c}^{c+\varepsilon} (y - c)\, f(y^\lambda)\, \sigma(y)\, dy < \infty \implies \\
&\mathsf{P}\Big\{\exists \varepsilon = \varepsilon(\omega) > 0 : \int_0^{\varepsilon} f\big((c+V_s)^\lambda\big)\, \sigma(c+V_s)\, ds < \infty\Big\} = 1 \implies \\
&\mathsf{P}\Big\{\exists \varepsilon = \varepsilon(\omega) > 0 : \int_0^{S_\varepsilon(V)} f\big((c+V_s)^\lambda\big)\, \sigma(c+V_s)\, ds < \infty\Big\} = 1 \implies \\
&\mathsf{P}\Big\{\exists \alpha = \alpha(\omega) < \gamma : \int_{T_\alpha(\rho)}^{T_\gamma(\rho)} f(\rho_s)\, ds < \infty\Big\} = 1 \implies \\
&\mathsf{P}\Big\{\exists \varepsilon = \varepsilon(\omega) > 0 : \int_{T_\gamma(\rho)-\varepsilon}^{T_\gamma(\rho)} f(\rho_s)\, ds < \infty\Big\} = 1.
\end{aligned}$$

This proves (2.4) for $\delta > 2$, $\gamma > \rho_0$. The other cases as well as (2.5) are treated in the similar manner. $\square$

`Proof of Theorem 2.6.` We will give the proof only for (2.6). We can write $\rho_t = B^{\lambda}_{\tau_t}$, where B_t starts at $x_0 = \rho_0^{2-\delta}$. Property (1.7) of the Bessel processes implies that

$$\lim_{t\to\infty} B_{\tau_t} = \lim_{t\to\infty} \rho_t^{2-\delta} = 0 \quad \text{a.s.}$$

Therefore, $\tau_\infty = T_0(B)$ a.s. Set $\bar{f}(x) = f(x)I(x > \rho_0)$. We have

$$\int_0^\infty \bar{f}(\rho_s)\,ds \overset{\text{a.s.}}{=} \int_0^{T_0(B)} \bar{f}(B_s^\lambda)\,\sigma(B_s)\,ds \overset{\text{law}}{=} \int_0^{S_{x_0}(V)} \bar{f}(V_s^\lambda)\,\sigma(V_s)\,ds,$$

where V_t is the three-dimensional Bessel process started at zero. We may assume that $\rho_0 \in C_g$ (otherwise, $\xi = 0$). Then the following implications are valid:

$$\begin{aligned}
&\int_{\rho_0}^\infty x\,f(x)\,dx < \infty \implies \int_0^\infty x\,\bar{f}(x)\,dx < \infty \implies \\
&\int_0^\infty y\,\bar{f}(y^\lambda)\,\sigma(y)\,dy < \infty \implies \\
&\mathsf{P}\Big\{\int_0^{S_{x_0}(V)} \bar{f}(V_s^\lambda)\,\sigma(V_s)\,ds < \infty\Big\} = 1 \implies \\
&\mathsf{P}\Big\{\int_0^\infty \bar{f}(\rho_s)\,ds < \infty\Big\} = 1.
\end{aligned}$$

Now, statement (2.6) follows from the fact that

$$\int_0^\infty \bar{f}(\rho_s)\,ds < \infty \implies \int_0^\infty f(\rho_s)\,ds < \infty$$

on the set $\{\xi < \infty\}$. Statement (2.7) as well as (ii) are proved in the same way. $\square$

`Proof of Theorem 2.7.` We will give the proof for $0 < \delta < 2$. We may assume that $\rho_t = B^\lambda_{\tau_t}$. The process A_t which is defined in Lemma 3.1 (i) satisfies the following inequality: $A_t \le \int_0^t \sigma(|B_s|)\,ds$. The function σ is locally integrable on $[0,\infty)$. The Engelbert-Schmidt Zero-One law (Proposition 1.1) implies that $\int_0^t \sigma(|B_s|)\,ds < \infty$ a.s. Thus, for each $t > 0$, we have $A_t < \infty$ a.s., and consequently, $\tau_t \xrightarrow[t\to\infty]{\text{a.s.}} \infty$. We arrive at the equality

$$\int_0^\infty f(\rho_s)\,ds \overset{\text{a.s.}}{=} \int_0^\infty f(B_s^\lambda)\sigma(B_s)\,ds,$$

The desired result now follows from the well-known property of the Brownian motion (see [14, ch. X, Proposition 3.11]):

$$\int_{\mathbb{R}} h(x)\,dx > 0 \implies \int_0^\infty h(B_s)\,ds = \infty \quad \text{a.s.}$$

$\square$

4 The Application of the Obtained Results

Let P_a^δ denote the distribution of the δ-dimensional ($\delta \geq 0$) Bessel process started at $a \geq 0$ (i.e. P_a^δ is a probability measure on the path space $C([0,\infty))$).

The following theorem is the main result of this section.

Theorem 4.1. *Suppose that $\delta \geq 0$, $\eta \geq 0$ and $\eta \neq \delta$. Then*

(i) *for each $a \geq 0$, the measures P_a^η and P_a^δ are singular on the σ-field $\mathcal{F}_\infty = \sigma(X_s,\, s \geq 0)$, where X_t denotes the coordinate process on $C([0,\infty))$;*

(ii) *the measures P_0^η and P_0^δ are singular on the σ-field $\mathcal{F}_{0+} = \bigcap_{\varepsilon>0}\sigma(X_s,\, s \leq \varepsilon)$.*

The proof of Theorem 4.1 is based on the following statement.

Lemma 4.2. *Let ρ_t be the δ-dimensional Bessel process started at $\rho_0 \geq 0$. If $\delta \geq 2$, then*

$$\frac{1}{\ln t}\int_1^t \rho_s^{-2}\, ds \xrightarrow[t\to\infty]{a.s.} \frac{1}{\delta-2}.$$

`Proof`. Let us first suppose that $\rho_0 = 0$. Then the process ρ_t satisfies the following self-similarity property:

$$\forall c > 0, \quad \mathrm{Law}\left(\frac{1}{\sqrt{c}}\rho_{ct},\, t \geq 0\right) = \mathrm{Law}(\rho_t,\, t \geq 0). \tag{4.1}$$

In order to prove (4.1), it is sufficient to note that the process $(\frac{1}{c}\rho_{ct}^2, t \geq 0)$ satisfies SDE (1.1), and therefore, it has the same distribution as the process $(\rho_t^2,\, t \geq 0)$.

It follows from (4.1) that the sequence

$$\zeta_k = \int_{e^k}^{e^{k+1}} \rho_s^{-2}\, ds, \quad k = 0, 1, \ldots$$

is stationary. According to Birkhoff's theorem,

$$\frac{1}{n}\sum_{k=0}^{n-1}\zeta_k = \frac{1}{n}\int_1^{e^n}\rho_s^{-2}\, ds \xrightarrow[n\to\infty]{\text{a.s.}} \mathsf{E}[\zeta_1|\mathcal{G}],$$

where $\mathcal{G}$ denotes the σ-field of invariant sets. Any set from $\mathcal{G}$ belongs to the σ-field $\bigcap_{n\in\mathbb{N}}\sigma(\zeta_n, \zeta_{n+1}, \ldots)$ which, in turn, belongs to the tail σ-field $\mathcal{X} = \bigcap_{t>0}\sigma(\rho_s,\, s \geq t)$.

Let us now prove that $\mathcal{X}$ is trivial, i.e. the probability of any set from $\mathcal{X}$ equals either 0 or 1. The Bessel process is a Feller process (see [14, ch. XI, section 1]), and therefore, it satisfies Blumenthal's Zero-One law. In other words, the σ-field $\mathcal{F}_{0+}^\rho = \bigcap_{\varepsilon>0}\sigma(\rho_s,\, s \leq \varepsilon)$ is trivial. Moreover, the Bessel process started at zero has the following time-inversion property (see [15]):

$$\mathrm{Law}(t\rho_{1/t},\, t > 0) = \mathrm{Law}(\rho_t,\, t > 0). \tag{4.2}$$

The triviality of $\mathcal{F}_{0+}^\rho$, together with (4.2), implies that $\mathcal{X}$ is also trivial.

Thus,

$$\frac{1}{n}\int_1^{e^n}\rho_s^{-2}\, ds \xrightarrow[n\to\infty]{\text{a.s.}} \mathsf{E}\left[\int_1^e \rho_s^{-2}\, ds\right].$$

Since the process ρ_t^{-2} is positive, we arrive at:

$$\frac{1}{\ln t}\int_1^t \rho_s^{-2}\,ds \xrightarrow[t\to\infty]{\text{a.s.}} \mathsf{E}\left[\int_1^e \rho_s^{-2}\,ds\right]. \tag{4.3}$$

In order to compute the constant in (4.3), we use the fact that the distribution of the random variable ρ_t^2 has the following density (see [14, ch. XI, Corollary 1.4]):

$$p(x) = (2t)^{-\delta/2}\,\Gamma(\delta/2)^{-1}\,y^{\delta/2-1}\,\exp(-y/2t),$$

where Γ denotes the gamma-function. Easy computations show that

$$\mathsf{E}\left[\int_1^e \rho_s^{-2}\,ds\right] = \frac{1}{\delta-2}.$$

Thus, the Lemma is proved for $\rho_0 = 0$.

If ρ_t is the Bessel process started at zero and $a > 0$, then $T_a(\rho) < \infty$ a.s. (see (1.7), (1.8)). It follows from the above reasoning that

$$\frac{1}{\ln t}\int_1^t \rho_{T_a(\rho)+s}^{-2}\,ds = \frac{1}{\ln t}\int_{T_a(\rho)+1}^{T_a(\rho)+t} \rho_s^{-2}\,ds \xrightarrow[t\to\infty]{\text{a.s.}} \frac{1}{\delta-2}. \tag{4.4}$$

By the strong Markov property,

$$\mathrm{Law}(\rho_{T_a(\rho)+t},\, t\ge 0) = \mathrm{Law}(\widetilde{\rho}_t,\, t\ge 0),$$

where $\widetilde{\rho}_t$ is the δ-dimensional Bessel process started at a. Formula (4.4) proves the Lemma in the general case. □

`Proof of Theorem 4.1.` **(i)** We will consider the case $a > 0$ (for $a = 0$, (i) follows from (ii)).

For $\delta = 0$, the statement is obvious since zero is an absorbing point for the zero-dimensional Bessel process.

Now, let $0 < \delta < \eta < 2$. Then $\mathrm{P}_a^\delta\{T_0(X) < \infty\} = \mathrm{P}_a^\eta\{T_0(X) < \infty\} = 1$ (see (1.6)). Let us choose d satisfying the inequality $-\eta < d < -\delta$. Corollary 2.4, in view of the strong Markov property, implies that

$$\int_{T_0(X)}^{T_0(X)+1} X_s^d\,ds = \infty \quad \mathrm{P}_a^\delta\text{-a.s.}, \tag{4.5}$$

$$\int_{T_0(X)}^{T_0(X)+1} X_s^d\,ds < \infty \quad \mathrm{P}_a^\eta\text{-a.s.} \tag{4.6}$$

This yields the singularity of P_a^η and P_a^δ on $\mathcal{F}_\infty$.

For $0 < \delta < 2 \le \eta$, the desired assertion is a consequence of the equalities: $\mathrm{P}_a^\delta\{T_0(X) < \infty\} = 1$ and $\mathrm{P}_a^\eta\{T_0(X) < \infty\} = 0$.

If $2 \le \delta < \eta$, then the statement of Theorem 4.1 follows from Lemma 4.2.

(ii) For $\delta = 0$, the statement is obvious.

If $0 < \delta < 2$ and $\delta < \eta$, we can choose d satisfying the inequality $-(\eta\wedge 2) < d < -\delta$, and the desired result follows from (4.5), (4.6) (in this case, $T_0(X) = 0$).

We now turn to the case $2 \le \delta < \eta$. Let ρ_t be the δ-dimensional Bessel process. Set $\widetilde{\rho}_t = t\rho_{1/t}$. Due to the time-inversion property (see (4.2)), the process $(\widetilde{\rho}_t,\, t \ge 0)$ has the same distribution as $(\rho_t,\, t \ge 0)$. By Lemma 4.1, for any $n \in \mathbb{N}$, we have

$$\frac{1}{\ln t}\int_{1/t}^{1/n} \rho_s^{-2}\, ds = \frac{1}{\ln t}\int_n^t s^{-2}\,\rho_{1/s}^{-2}\, ds = \frac{1}{\ln t}\int_n^t \widetilde{\rho}_s^{-2}\, ds \xrightarrow[t\to\infty]{\text{a.s.}} \frac{1}{\delta-2}.$$

Hence, the set

$$A_n = \left\{X \in C([0,\infty)) : \frac{1}{\ln t}\int_{1/t}^{1/n} X_s^{-2}\, ds \xrightarrow[t\to\infty]{} \frac{1}{\delta-2}\right\}$$

has P_0^δ-measure 1. Similarly, $\mathrm{P}_0^\eta(B_n) = 1$ for

$$B_n = \left\{X \in C([0,\infty)) : \frac{1}{\ln t}\int_{1/t}^{1/n} X_s^{-2}\, ds \xrightarrow[t\to\infty]{} \frac{1}{\eta-2}\right\}.$$

Consequently, for the sets $A = \liminf_{n\to\infty} A_n$ and $B = \liminf_{n\to\infty} B_n$, we have: $\mathrm{P}_0^\delta(A) = 1$ and $\mathrm{P}_0^\eta(B) = 1$. Moreover, A and B belong to the σ-field $\mathcal{F}_{0+}$. Besides, $A \cap B = \emptyset$. Thus, the measures P_0^δ and P_0^η are singular on $\mathcal{F}_{0+}$. This completes the proof. □

`Remarks`. (i) If $a > 0$ and $\delta \wedge \eta \ge 2$, then P_a^η and P_a^δ are equivalent on each σ-field $\mathcal{F}_t = \sigma(X_s,\, s \le t)$.

(ii) If $a > 0$ and $\delta \wedge \eta < 2$, then P_a^η and P_a^δ are neither equivalent nor singular on $\mathcal{F}_t$, $t > 0$. □

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