# GENERAL ARBITRAGE PRICING MODEL: POSSIBILITY APPROACH

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**Abstract.** We introduce the *possibility approach* to pricing by arbitrage. The characteristic feature of this approach is that it does not employ the historic probability measure.

The study is performed on two levels of generality:

- for a static model with a finite number of assets;
- for a general arbitrage pricing model introduced in [3].

The main results obtained for each of these models are: the fundamental theorem of asset pricing and the representation of the fair price intervals.

**Key words and phrases.** Fair price, fundamental theorem of asset pricing, general arbitrage pricing model, generalized arbitrage, possibility space, risk-neutral measure, set of attainable incomes, set of possible elementary events, transaction costs.

# 1 Introduction

**1. Purpose of the paper.** When a coin is tossed, everyone agrees that there exists a probability measure on the set of elementary outcomes, and this measure assigns the mass $1/2$ to each of the two outcomes. When shooting at a target is performed, everyone agrees that there exists a probability measure on the set of elementary outcomes. The exact form of this measure cannot be found by pure thought, but can be estimated by repeating the trials. In both examples, the legitimacy of a probability measure is based on the existence of a fixed set of conditions that admits an unlimited number of repetitions. The importance of such a set of conditions was stressed by Kolmogorov [7; Ch. I, § 2].

In the problems that finance deals with, such a fixed set of conditions does not seem to exist at all. Therefore, it is questionable whether there exists a historic measure $\mathsf{P}$, which serves as an input to the overwhelming majority of arbitrage pricing models. It is unquestionable that even if such a measure exists, then no one knows exactly what it is.

But let us now recall that the origin of arbitrage pricing lies in decomposing a complicated contract into simpler contracts, and this does not require any probability considerations. Another example: when calculating the exchange rate through the triangular arbitrage, the probability measure is not needed. One more example: the trivial interval $((S_0 - K)^+, S_0)$ of fair prices of a European call option is obtained with no probability at all. However, in more complicated models the structure of $\mathsf{P}$ is essential. The basic example in this line is the Black–Scholes model, in which it is the particular structure of $\mathsf{P}$ that yields the completeness. Of course, if $\mathsf{P}$ is eliminated in such a model, then we get unacceptably wide intervals of fair prices. But nevertheless, in some cases the following effect takes place: if the market prices of a sufficient number of traded derivatives are taken into account, then one can obtain a reasonably small fair price interval of a new contract without relying on the original probability measure. This method was successfully employed (for various models) in the papers [1], [2], [6].

The above observation justifies the general possibility approach to arbitrage pricing. It requires as the first input the set of all possible outcomes and does not require the probabilities assigned to these outcomes. To be more precise, the possibility approach is based on a *possibility space* $(\Omega, \mathcal{F})$ instead of a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. We call $\Omega$ the *set of possible elementary events*. Usually it can be defined by pure thought (i.e. without using the real data) in an unambiguous way. For example, a natural set of possible prices of an equity is $\mathbb{R}_{++} (= (0, \infty))$; a natural set of possible prices of $d$ equities is $\mathbb{R}^d_{++}$. Typically, the set of possible elementary events admits a natural topology, and $\mathcal{F}$ is taken as its Borel $\sigma$-field.

The possibility approach is introduced on two levels of generality: first, we consider a static model with a finite number of assets, and then we consider the general arbitrage pricing model introduced in [3].

**2. Static model with a finite number of assets.** This is a classical model of financial mathematics (a review of arbitrage pricing in this model can be found, for example, in [5; Ch. 1] or [3; Sect. 2]). In Section 2, we consider the possibility version of this model.

We introduce the possibility variant of the No Arbitrage (NA) condition and prove that this condition is satisfied if and only if for any nonempty set $D \in \mathcal{F}$, there exists a martingale measure $\mathsf{Q}$ such that $\mathsf{Q}(D) > 0$. A geometric criterion is presented as well.

Furthermore, using the possibility version of the NA condition, we define the set of fair prices of a contingent claim $F$ and prove that it coincides up to endpoints with the interval $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{M}\}$, where $\mathcal{M}$ denotes the set of martingale measures. A geometric representation of this set is given as well.

**3. General arbitrage pricing model.** In [3], we presented a unified approach to pricing contingent claims through a new concept of *generalized arbitrage*. The No Generalized Arbitrage (NGA) condition is a strengthening of the classical NA condition. This was done within the framework of a *general arbitrage pricing model*. Various models of arbitrage pricing theory, including

- static as well as dynamic models;
- models with an infinite number of assets;
- models with transaction costs (see [4]),

can be viewed as particular cases of this general model.

In Section 3, we consider the possibility version of a general arbitrage pricing model. It is defined as a triple $(\Omega, \mathcal{F}, A)$, where $(\Omega, \mathcal{F})$ is a possibility space and $A$ is a convex cone in the space of all $\mathcal{F}$-measurable real-valued functions. From the financial point of view, $A$ is the set of discounted incomes that can be obtained in the model under consideration.

For a model $(\Omega, \mathcal{F}, A)$, we introduce the possibility variant of the NGA condition. Similarly to [3], we define a *risk-neutral measure* as a measure $\mathsf{Q}$ on $\mathcal{F}$ such that $\mathsf{E}_\mathsf{Q} X^- \geq \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$ ($X^-$ and $X^+$ denote the negative part and the positive part of $X$, respectively; the expectations $\mathsf{E}_\mathsf{Q} X^-$, $\mathsf{E}_\mathsf{Q} X^+$ here are allowed to take on the value $+\infty$).

Theorem 3.6 states (under a natural assumption) that the NGA condition is satisfied if and only if for any nonempty set $D \in \mathcal{F}$, there exists a risk-neutral measure $\mathsf{Q}$ such that $\mathsf{Q}(D) > 0$. Thus, a risk-neutral measure appears to be a more fundamental object than a historic probability measure. A nice illustration is provided by the bookmaking, where the "true" distribution on the set of outcomes is completely unclear, while the "market-estimated" distribution is easily recovered from the bets.

Next we consider the problem of pricing contingent claims. We define a fair price of a contingent claim $F$ ($F$ is a measurable function on $(\Omega, \mathcal{F})$) as a real number $x$ such that the extended model $(\Omega, \mathcal{F}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGA condition. Theorem 3.9 states (under some natural assumptions) that the set of fair prices of $F$ coincides up to endpoints with the interval $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\}$, where $\mathcal{R}$ denotes the set of risk-neutral measures.

**4. Particular models.** In order to apply the general results of Section 3 to a particular model, one should

**1.** specify the set $A$ of attainable incomes (this is typically done in a straightforward way);

**2.** find out the structure of the set of risk-neutral measures (typically, the risk-neutral measures in a particular model admit a simpler description than the general definition of a risk-neutral measure).

Once this is done, Theorem 3.6 gives the necessary and sufficient conditions for the absence of generalized arbitrage, while Theorem 3.9 yields the form of the set of fair prices of a contingent claim. Both procedures 1 and 2 were implemented in [3], [4] for a number of particular models.

However, the possibility framework gives rise to an interesting question: Is the NGA condition (in its possibility version) satisfied in a particular model? The answer depends on the "geometry" of the price structure. In Sections 4–5, we study this problem for a number of particular models, namely

- a discrete-time model with a finite number of assets (Section 4);
- a continuous-time model with a finite number of assets (Section 5);
- a model with European call options as basic assets (Section 6).

***Acknowledgement.*** I am thankful to the anonymous referee for a very careful reading of the manuscript and important suggestions.

# 2 Static Model with Finite Number of Assets

The reader is invited to compare this section with [3; Sect. 2].

**Definition 2.1.** A *possibility space* is a pair $(\Omega, \mathcal{F})$, where $\Omega$ is a set and $\mathcal{F}$ is a $\sigma$-field on $\Omega$. We call $\Omega$ the *set of possible elementary events.*

Let $(\Omega, \mathcal{F})$ be a possibility space. Let $S_0 \in \mathbb{R}^d$ and $S_1$ be an $\mathbb{R}^d$-valued $\mathcal{F}$-measurable function. From the financial point of view, $S_n^i$ is the discounted price of the $i$-th asset at time $n$. Define the set of attainable incomes by

$$A = \left\{ \sum_{i=1}^{d} h^i (S_1^i - S_0^i) : h^i \in \mathbb{R} \right\}.$$

**Definition 2.2.** A model $(\Omega, \mathcal{F}, S_0, S_1)$ satisfies the *No Arbitrage* (NA) condition if $A \cap L_+^0 = \{0\}$ ($L_+^0$ denotes the set of $\mathbb{R}_+$-valued $\mathcal{F}$-measurable functions).

**Definition 2.3.** A *martingale measure* is a probability measure $\mathsf{Q}$ on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}}|S_1| < \infty$ and $\mathsf{E}_{\mathsf{Q}} S_1 = S_0$. The set of martingale measures will be denoted by $\mathcal{M}$.

Notation. Set $C = \overline{\operatorname{conv}}\{S_1(\omega) : \omega \in \Omega\}$ and let $C^\circ$ denote the relative interior of $C$.

**Theorem 2.4 (Fundamental theorem of asset pricing).** *For the model* $(\Omega, \mathcal{F}, S_0, S_1)$, *the following conditions are equivalent:*

(a) *NA*;
(b) $S_0 \in C^\circ$;
(c) *for any* $D \in \mathcal{F} \setminus \{\emptyset\}$, *there exists* $\mathsf{Q} \in \mathcal{M}$ *such that* $\mathsf{Q}(D) > 0$.

Proof. *Step 1.* Let us prove the implication (a)$\Rightarrow$(b). If $S_0 \notin C^\circ$, then, by the separation theorem, there exists $h \in \mathbb{R}^d$ such that $\langle h, (S_1 - S_0)\rangle \geq 0$ pointwise and $\langle h, (S_1(\omega) - S_0(\omega))\rangle > 0$ for some $\omega \in \Omega$. This contradicts the NA condition.

*Step 2.* Let us prove the implication (b)$\Rightarrow$(c). Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. The set

$$E = \left\{ \sum_{k=0}^{m} \alpha_k S_1(\omega_k) : m \in \mathbb{N},\ \omega_1, \dots, \omega_m \in \Omega,\ \alpha_0, \dots, \alpha_m \in \mathbb{R}_{++},\ \sum_{k=0}^{m} \alpha_k = 1 \right\}$$

is convex, and the closure of $E$ contains $\{S_1(\omega) : \omega \in \Omega\}$. Consequently, $E \supseteq C^\circ$. Thus, there exist $\omega_1, \dots, \omega_m \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $\sum_{k=0}^m \alpha_k = 1$ and $\sum_{k=0}^m \alpha_k S_1(\omega_k) = S_0$. Then the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$ belongs to $\mathcal{M}$ and $\mathsf{Q}(D) > 0$ ($\delta_\omega$ denotes the point mass concentrated on $\{\omega\}$, i.e. $\delta_\omega(A) = I(\omega \in A)$).

*Step 3.* Let us prove the implication (c)$\Rightarrow$(a). Suppose that the NA condition is not satisfied, i.e. there exists $X \in A \cap (L_+^0 \setminus \{0\})$. Consider $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(X > 0) > 0$. Then $\mathsf{E}_{\mathsf{Q}} X > 0$. On the other hand, as $\mathsf{Q} \in \mathcal{M}$, we should have $\mathsf{E}_{\mathsf{Q}} X = 0$. The obtained contradiction shows that the NA condition is satisfied. □

Now, let $F$ be a real-valued $\mathcal{F}$-measurable function. From the financial point of view, $F$ is the discounted payoff of some contingent claim.

**Definition 2.5.** A real number $x$ is a *fair price* of $F$ if the model with $d+1$ assets $(\Omega, \mathcal{F}, x, S_0^1, \dots, S_0^d, F, S_1^1, \dots, S_1^d)$ satisfies the NA condition. The set of fair prices of $F$ will be denoted by $I(F)$.

Notation. Set $D = \overline{\operatorname{conv}}\{(F(\omega), S_1(\omega)) : \omega \in \Omega\}$ and let $D^\circ$ denote the relative interior of $D$.

For two subsets $I$, $J$ of the real line, by the notation $I \approx J$ we will mean that the interiors of $I$ and $J$ coincide and the closures of $I$ and $J$ coincide. In particular, if $I \approx J$ and $J$ is an interval (that may be closed, open, or semi-open), then $I$ is also an interval, and $I$ coincides with $J$ up to the endpoints.

**Theorem 2.6.** *Suppose that the model* $(\Omega, \mathcal{F}, S_0, S_1)$ *satisfies the NA condition. Then*

$$I(F) = \{x : (x, S_0) \in D^\circ\} \approx \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\}. \tag{2.1}$$

*The expectation* $\mathsf{E}_{\mathsf{Q}} F$ *here is taken in the sense of finite expectations, i.e. we consider only those* $\mathsf{Q}$*, for which* $\mathsf{E}_{\mathsf{Q}}|F| < \infty$.

`Proof`. Theorem 2.4 implies that

$$I(F) = \{x : (x, S_0) \in D^\circ\} \subseteq \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\}. \tag{2.2}$$

Let $x \in \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\}$. Take $\mathsf{Q}_0 \in \mathcal{M}$ such that $x = \mathsf{E}_{\mathsf{Q}_0} F$. One can find $\mathsf{Q}_1 \in \mathcal{M}$ such that $\mathsf{E}_{\mathsf{Q}_1}|F| < \infty$ and $\overline{\mathrm{conv}}\, \mathrm{supp}\, \mathrm{Law}_{\mathsf{Q}_1}(F, S_1) = D$ ($\mathsf{Q}_1$ can be found in the form $\sum_{n=1}^{\infty} \alpha_n \delta_{\omega_n}$). For any $\varepsilon \in (0,1)$, the measure $\mathsf{Q}(\varepsilon) = (1-\varepsilon)\mathsf{Q}_0 + \varepsilon \mathsf{Q}_1$ belongs to $\mathcal{M}$ and $\overline{\mathrm{conv}}\, \mathrm{supp}\, \mathrm{Law}_{\mathsf{Q}(\varepsilon)}(F, S_1) = D$. Therefore, $\mathsf{E}_{\mathsf{Q}(\varepsilon)}(F, S_1) \in D^\circ$, which means that

$$\mathsf{E}_{\mathsf{Q}(\varepsilon)} F \in \{x : (x, S_0) \in D^\circ\}.$$

Furthermore, $\mathsf{E}_{\mathsf{Q}(\varepsilon)} F \xrightarrow[\varepsilon \downarrow 0]{} x$. This, together with (2.2), proves the approximate equality in (2.1). □

`Remarks`. `(i)` Let $V_*(F)$ (resp., $V^*(F)$) denote the left (resp., right) endpoint of $I(F)$. Let $F$ be such that $V_*(F) < V^*(F)$. It follows from the equality $I(F) = \{x : (x, S_0) \in D^\circ\}$ that $I(F) = (V_*(F), V^*(F))$. As for the interval $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\}$, it has the endpoints $V_*(F)$ and $V^*(F)$, but may contain them. For instance, this interval contains $V^*(F)$ if and only if

$$(V^*(F), S_0) \in \mathrm{conv}\{(F(\omega), S_1(\omega)) : \omega \in \Omega\}.$$

`(ii)` Another way to define the fair price interval could be as follows. We introduce the lower and the upper prices as

$$V_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \text{ pointwise}\},$$
$$V^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \text{ pointwise}\},$$

and the fair price interval is defined as the interval with the endpoints $V_*(F)$ and $V^*(F)$. Using the equality $I(F) = \{x : (x, S_0) \in D^\circ\}$ and elementary geometric considerations, one can check that if the model $(\Omega, \mathcal{F}, S_0, S_1)$ satisfies the NA condition, then the values $V_*(F)$ and $V^*(F)$ defined this way coincide with the values defined in the previous remark.

# 3 General Arbitrage Pricing Model

The reader is invited to compare this section with [3; Sect. 3].

**Definition 3.1.** A *general arbitrage pricing model* is a triple $(\Omega, \mathcal{F}, A)$, where $(\Omega, \mathcal{F})$ is a possibility space and $A$ is a convex cone in $L^0$ ($L^0$ is the space of real-valued $\mathcal{F}$-measurable functions). The set $A$ will be called the *set of attainable incomes.*

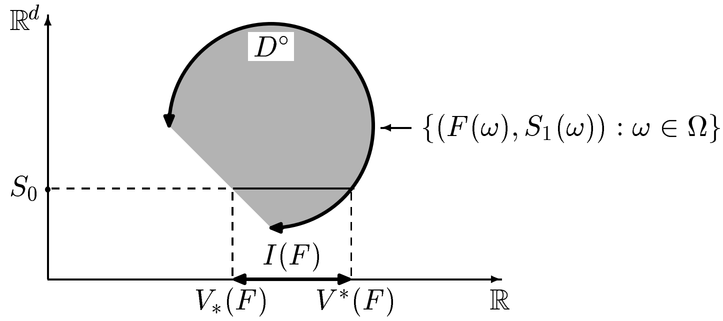


Figure 1. The joint arrangement of $I(F)$, $V_*(F)$, $V^*(F)$, $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{M}\}$, and $D^\circ$. In the example shown here, $I(F) = (V_*(F), V^*(F))$, while $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{M}\} = (V_*(F), V^*(F)]$.

`Notation`. (i) Set

$$B = \Big\{ Z \in L^0 : \text{ there exist } (X_n)_{n\in\mathbb{N}} \in A \text{ and } a \in \mathbb{R} \text{ such that } X_n \geq a \text{ pointwise and } Z = \lim_{n\to\infty} X_n \text{ pointwise} \Big\}. \tag{3.1}$$

(ii) For $Z \in B$, denote $\gamma(Z) = 1 - \inf_{\omega\in\Omega} Z(\omega)$ and set

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L^0_+\}, \\ A_2(Z) &= \left\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \right\}, \\ A_3(Z) &= A_2(Z) \cap L^\infty, \\ A_4(Z) &= \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, M_F). \end{aligned} \tag{3.2}$$

Here $L^0_+$ is the set of $\mathbb{R}_+$-valued elements of $L^0$; $L^\infty$ is the space of bounded elements of $L^0$; $\sigma(L^\infty, M_F)$ denotes the weak topology on $L^\infty$ induced by the space $M_F$ of finite $\sigma$-additive measures on $\mathcal{F}$ (i.e. signed measures with finite variation).

**Definition 3.2.** The model $(\Omega, \mathcal{F}, A)$ satisfies the *No Generalized Arbitrage* (NGA) condition if for any $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.

**Definition 3.3.** A *risk-neutral measure* is a probability measure $\mathsf{Q}$ on $\mathcal{F}$ such that $\mathsf{E}_\mathsf{Q} X^- \geq \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$. The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of risk-neutral measures will be denoted by $\mathcal{R}$.

`Notation`. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of probability measures $\mathsf{Q}$ on $\mathcal{F}$ with the property: for any $X \in A$ such that $X \geq -\alpha Z - \beta$ pointwise with some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \leq 0$.

The following lemma is almost the same as [3; Lem. 3.4].

**Lemma 3.4.** *For any $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

The following basic assumption is satisfied in all the particular models considered below.

**Assumption 3.5.** There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (in particular, both sets might be empty).

**Theorem 3.6 (Fundamental theorem of asset pricing).** *Suppose that Assumption 3.5 is satisfied. Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition if and only if for any* $D \in \mathcal{F} \setminus \{\emptyset\}$, *there exists a risk-neutral measure* $\mathsf{Q}$ *such that* $\mathsf{Q}(D) > 0$.

The proof of Theorem 3.6 follows the same lines as the proof of its probabilistic analog in [3; Th. 3.6]. It is based on the following possibility analog of the Kreps–Yan theorem:

**Lemma 3.7.** *Let* $C$ *be a* $\sigma(L^\infty, M_F)$*-closed convex cone in* $L^\infty$ *such that* $C \supseteq L^\infty_-$ ($L^\infty_-$ *is the set of negative elements of* $L^\infty$). *Let* $W \in L^\infty \setminus C$. *Then there exists a probability measure* $\mathsf{Q}$ *on* $\mathcal{F}$ *such that* $\mathsf{E_Q} X \le 0$ *for any* $X \in C$ *and* $\mathsf{E_Q} W > 0$.

Proof. By the Hahn-Banach separation theorem (see [9; Ch. II, Th. 9.2]), there exists a measure $\mathsf{Q}_0 \in M_F$ such that $\mathsf{E}_{\mathsf{Q}_0} W \notin \{\mathsf{E}_{\mathsf{Q}_0} X : X \in C\}$. Without loss of generality, $\mathsf{E}_{\mathsf{Q}_0} W > 0$. As $C$ is a cone, $\mathsf{E}_{\mathsf{Q}_0} X \le 0$ for any $X \in C$. Since $C \supseteq L^\infty_-$, $\mathsf{Q}_0$ is positive. Then the measure $\mathsf{Q} = c\mathsf{Q}_0$, where $c$ is the normalizing constant, satisfies the desired properties. □

Proof of Theorem 3.6. *Step 1.* Let us prove the "only if" implication. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Set $W = I_D$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 3.6 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4(Z_0)$ and to the point $W$ yields a probability measure $\mathsf{Q}_0$ on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \le 0$ for any $X \in A_4(Z_0)$ and $\mathsf{E}_{\mathsf{Q}_0} W > 0$. By the Fatou lemma, for any $X \in A$ such that $\frac{X}{Z_0 + \gamma(Z_0)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0 + \gamma(Z_0)} \le 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_0 + \gamma(Z_0)} \mathsf{Q}_0$, where $c$ is the normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$ and

$$\mathsf{Q}(D) = \mathsf{E}_{\mathsf{Q}_0} \frac{cW}{Z_0 + \gamma(Z_0)} > 0.$$

*Step 2.* Let us prove the "if" implication. Suppose that the NGA condition is not satisfied. Then there exist $Z \in B$ and $W \in A_4(Z) \cap (L^0_+ \setminus \{0\})$. Take $\mathsf{Q} \in \mathcal{R}$ such that $\mathsf{Q}(W > 0) > 0$. It follows from the Fatou lemma that $Z$ is $\mathsf{Q}$-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where $c$ is the normalizing constant. For any $X \in A$ such that $\frac{X}{Z + \gamma(Z)}$ is bounded below by a constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E_Q} X^- \le \mathsf{E_Q}(\alpha Z + \alpha\gamma(Z)) < \infty,$$

and consequently,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{X}{Z + \gamma(Z)} = c\mathsf{E_Q} X \le 0.$$

Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} X \le 0$ for any $X \in A_4(Z)$. On the other hand,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} W = c\mathsf{E_Q}(Z + \gamma(Z))W > 0.$$

The obtained contradiction shows that the NGA condition is satisfied. □

Remark. It is seen from the above proof that the necessity part in Theorem 3.6 is true without Assumption 3.5. It can be shown that this assumption is essential for the sufficiency part.

Now, let $F$ be an $\mathcal{F}$-measurable function meaning the discounted payoff of some contingent claim.

**Definition 3.8.** A real number $x$ is a *fair price* of $F$ if the extended model $(\Omega, \mathcal{F}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGA condition. The set of fair prices of $F$ will be denoted by $I(F)$.

**Theorem 3.9 (Pricing contingent claims).** *Suppose that the model* $(\Omega, \mathcal{F}, A)$ *satisfies Assumption 3.5 and the NGA condition, while* $F$ *is bounded below and* $\mathsf{E}_{\mathsf{Q}} F < \infty$ *for any* $\mathsf{Q} \in \mathcal{R}$*. Then*

$$I(F) \approx \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\}.$$

Proof. *Step 1.* Let us prove the inclusion

$$I(F) \subseteq \left[\inf_{\mathsf{Q} \in \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F,\ \sup_{\mathsf{Q} \in \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F\right].$$

Let $x \in I(F)$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Set $Z_1 = Z_0 + (F - x)$. Then $Z_1 \in B'$, where $B'$ is defined by (3.1) with $A$ replaced by

$$A' = \{X + h(F - x) : X \in A,\ h \in \mathbb{R}\}. \tag{3.3}$$

Set $W \equiv 1$. Lemma 3.7 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A'_4(Z_1)$ ($A'_4(Z_1)$ is defined by (3.2) with $A$ replaced by $A'$) and to the point $W$ yields a probability measure $\mathsf{Q}_0$ on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \le 0$ for any $X \in A'_4(Z_1)$. By the Fatou lemma, for any $X \in A'$ such that $\frac{X}{Z_1 + \gamma(Z_1)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_1 + \gamma(Z_1)} \le 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_1 + \gamma(Z_1)} \mathsf{Q}_0$, where $c$ is the normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_1) \subseteq \mathcal{R}(Z_0) = \mathcal{R}$. Moreover, $\mathsf{E}_{\mathsf{Q}}(x - F) \le 0$ and $\mathsf{E}_{\mathsf{Q}}(F - x) \le 0$ since the functions $\frac{x - F}{Z_1 + \gamma(Z_1)}$ and $\frac{F - x}{Z_1 + \gamma(Z_1)}$ are bounded below. Thus, $\mathsf{E}_{\mathsf{Q}} F = x$.

*Step 2.* Suppose that $\mathsf{E}_{\mathsf{Q}} F = \mathsf{E}_{\mathsf{Q}'} F$ for any $\mathsf{Q}, \mathsf{Q}' \in \mathcal{R}$. Let us prove the inclusion $\mathsf{E}_{\mathsf{Q}} F \in I(F)$. Denote $\mathsf{E}_{\mathsf{Q}} F$ by $x$. Suppose that $x \notin I(F)$, i.e. the model $(\Omega, \mathcal{F}, A')$, where $A'$ is given by (3.3), does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A'_4(Z) \cap (L^0_+ \setminus \{0\})$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 3.7 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4(Z_0)$ and to the point $W$ yields a probability measure $\mathsf{Q}_0$ on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \le 0$ for any $X \in A_4(Z_0)$ and $\mathsf{E}_{\mathsf{Q}_0} W > 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_0 + \gamma(Z_0)} \mathsf{Q}_0$, where $c$ is the normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$ and $\mathsf{E}_{\mathsf{Q}} W > 0$. Moreover, $\mathsf{E}_{\mathsf{Q}} F = x$.

Choose an arbitrary $Y = X + h(F - x) \in A'$ (here $X \in A$) such that $Y$ is bounded below. It follows from the condition $\mathsf{E}_{\mathsf{Q}} F = x$ that $\mathsf{E}_{\mathsf{Q}} X^- < \infty$. As $\mathsf{Q} \in \mathcal{R}$, we have $\mathsf{E}_{\mathsf{Q}} X \le 0$. This, combined with the condition $\mathsf{E}_{\mathsf{Q}} F = x$, implies that $\mathsf{E}_{\mathsf{Q}} Y \le 0$. By the Fatou lemma, $Z$ is $\mathsf{Q}$-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where $c$ is the normalizing constant. For any $Y = X + h(F - x) \in A'$ (here $X \in A$) such that $\frac{Y}{Z + \gamma(Z)}$ is bounded below by some constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_{\mathsf{Q}} Y^- \le \mathsf{E}_{\mathsf{Q}}(\alpha Z + \alpha\gamma(Z)) < \infty.$$

Consequently, $\mathsf{E}_{\mathsf{Q}} X^- < \infty$, $\mathsf{E}_{\mathsf{Q}} X \le 0$, and $\mathsf{E}_{\mathsf{Q}} Y \le 0$. This means that $\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{Y}{Z + \gamma(Z)} \le 0$. Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} W \le 0$. But this is a contradiction since $\widetilde{\mathsf{Q}} \sim \mathsf{Q}$ and $\mathsf{E}_{\mathsf{Q}} W > 0$. As a result, $x \in I(F)$.

*Step 3.* Let us prove the inclusion

$$\left(\inf_{\mathsf{Q} \in \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F,\ \sup_{\mathsf{Q} \in \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F\right) \subseteq I(F).$$

Let $x$ belong to the left-hand side of this inclusion, i.e.

$$\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_{\mathsf{Q}} F < x < \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_{\mathsf{Q}} F.$$

Suppose that $x \notin I(F)$, i.e. the model $(\Omega, \mathcal{F}, A')$, where $A'$ is defined by (3.3), does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A'_4(Z) \cap (L^0_+ \setminus \{0\})$. Applying the same reasoning as in the previous step, we find a measure $\mathsf{Q}_1 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_1} W > 0$. By the conditions of the theorem, $\mathsf{E}_{\mathsf{Q}_1}|F| < \infty$. Find measures $\mathsf{Q}_2, \mathsf{Q}_3 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_2} F < x$, $\mathsf{E}_{\mathsf{Q}_3} F > x$. Clearly, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_{++}$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\mathsf{E}_{\mathsf{Q}} F = x$, where $\mathsf{Q} = \alpha_1 \mathsf{Q}_1 + \alpha_2 \mathsf{Q}_2 + \alpha_3 \mathsf{Q}_3$. Note that $\mathsf{Q} \in \mathcal{R}$ due to the convexity of $\mathcal{R}$ and $\mathsf{E}_{\mathsf{Q}} W > 0$. The proof is now completed in the same way as in the previous step. □

The following example shows that the equality $I(F) = \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\}$ (which is true in the probability setting; see [3; Th. 3.10]) can be violated.

**Example 3.10.** Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $A = \{0\}$. Consider $F(\omega) = \omega$. Then $I(F) = (0,1)$, while $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\} = [0,1]$. □

The next example shows that the assumption "$\mathsf{E}_{\mathsf{Q}} F < \infty$ for any $\mathsf{Q} \in \mathcal{R}$" in Theorem 3.9 is essential.

**Example 3.11.** Let $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, and

$$A = \left\{ \sum_{n=1}^{N} h_n X_{a_n b_n} : N \in \mathbb{N},\ a_n < b_n \in \mathbb{R}_+,\ h_n \in \mathbb{R} \right\},$$

where

$$X_{ab}(\omega) = I(a < \omega \le b) - \int_a^b e^{-x} dx, \quad \omega \in \Omega.$$

Consider $F(\omega) = e^{\omega}$.

If $\mathsf{Q} \in \mathcal{R}$, then, for any $a > 0$, we have $\mathsf{E}_{\mathsf{Q}} X_{0a} = 0$ (note that $X_{0a}$ is bounded), which means that

$$\mathsf{Q}((0,a]) = \mathsf{Q}(\mathbb{R}_{++}) \int_0^a e^{-x} dx, \quad a \in \mathbb{R}_{++}.$$

Hence, $\mathsf{Q}$ has the form $\alpha_1 \mathsf{Q}_1 + \alpha_2 \mathsf{Q}_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_1 + \alpha_2 = 1$, $\mathsf{Q}_1 = \delta_0$, and $\mathsf{Q}_2$ is the exponential distribution on $\mathbb{R}_+$ with parameter 1. Clearly, any measure of this form belongs to $\mathcal{R}$. We have $\mathsf{E}_{\mathsf{Q}} F = 1$ if $\mathsf{Q} = \mathsf{Q}_1$ and $\mathsf{E}_{\mathsf{Q}} F = \infty$ otherwise. Consequently, $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\} = \{1\}$.

Take now $x \in I(F)$. For $n \in \mathbb{N}$, set

$$F_n(\omega) = \begin{cases} 0 & \text{if } \omega = 0, \\ e^m & \text{if } \omega \in (m, m+1],\ m = 0, \dots, n-1, \\ 0 & \text{if } \omega > n, \end{cases}$$

$$x_n = \int_0^\infty F_n(x) e^{-x} dx.$$

Then $F_n - x_n \in A$. As $x_n \to \infty$, there exists $n_0$ such that $x_{n_0} > x$. Then

$$(F(\omega) - x) - (F_{n_0}(\omega) - x_{n_0}) \ge x_{n_0} - x > 0, \quad \omega \in \Omega.$$

But

$$(F - x) - (F_n - x_n) \in A' = \{X + h(F - x) : X \in A,\ h \in \mathbb{R}\}.$$

This contradicts the choice of $x$. As a result, $I(F) = \emptyset$. □

# 4 Discrete-Time Model with Finite Number of Assets

We will consider a model with no transaction costs. Thus, we are given a possibility space $(\Omega, \mathcal{F})$ endowed with a filtration $(\mathcal{F}_n)_{n=0,\dots,N}$ and an $\mathbb{R}^d$-valued $(\mathcal{F}_n)$-adapted sequence $(S_n)_{n=0,\dots,N}$. From the financial point of view, $S_n^i$ is the discounted price of the $i$-th asset at time $n$. The set of attainable incomes is defined as

$$A = \left\{ \sum_{n=1}^{N} \sum_{i=1}^{d} H_n^i (S_n^i - S_{n-1}^i) : H_n \text{ is } \mathcal{F}_{n-1}\text{-measurable} \right\}.$$

We will assume that, for any $n = 0, \dots, N-1$, $\omega \in \Omega$, there exists an atom $\mathfrak{a}_n(\omega)$ of $\mathcal{F}_n$ that contains $\omega$. (Recall that an *atom* of a $\sigma$-field $\mathcal{F}$ is a set $\mathfrak{a} \in \mathcal{F}$ such that $\mathfrak{a} \neq \emptyset$ and, for any $D \in \mathcal{F}$, we have either $D \supseteq \mathfrak{a}$ or $D \cap \mathfrak{a} = \emptyset$.)

`Notation`. Set $C_n(\omega) = \overline{\operatorname{conv}}\{S_{n+1}(\omega') : \omega' \in \mathfrak{a}_n(\omega)\}$ and let $C_n^\circ(\omega)$ denote the relative interior of $C_n(\omega)$.

Let $\mathcal{M}$ denote the set of probability measures on $\mathcal{F}$, with respect to which $S$ is an $(\mathcal{F}_n)$-martingale.

**Theorem 4.1 (Fundamental theorem of asset pricing).** *For the model $(\Omega, \mathcal{F}, A)$, the following conditions are equivalent:*

(a) *NGA;*
(b) *NA (i.e. $A \cap L_+^0 = \{0\}$);*
(c) $S_n(\omega) \in C_n^\circ(\omega)$, $n = 0, \dots, N-1$, $\omega \in \Omega$;
(d) *for any $D \in \mathcal{F} \setminus \{\emptyset\}$, there exists $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(D) > 0$.*

**Lemma 4.2.** *Suppose that condition* (c) *of Theorem 4.1 is satisfied. Let $\omega_0 \in \Omega$. Then there exist $\omega_1, \dots, \omega_m \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $\sum_{k=0}^m \alpha_k = 1$ and $\sum_{k=0}^m \alpha_k X(\omega_k) = 0$ for any $X \in A$.*

`Proof`. We will prove this statement by the induction in $N$.

*Base of induction.* For $N = 1$, the statement is verified by the same arguments as those used in the proof of Theorem 2.4 (Step 2).

*Step of induction.* Assume that the statement is true for $N-1$. Let us prove it for $N$. By the induction hypothesis, there exist $\widetilde{\omega}_1, \dots, \widetilde{\omega}_l \in \Omega$ and $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_l \in \mathbb{R}_{++}$ such that $\widetilde{\omega}_0 = \omega_0$, $\sum_{i=0}^l \widetilde{\alpha}_i = 1$, and $\sum_{i=0}^l \widetilde{\alpha}_i X(\widetilde{\omega}_i) = 0$ for any $X \in A'$, where

$$A' = \left\{ \sum_{n=1}^{N-1} \sum_{i=1}^{d} H_n^i (S_n^i - S_{n-1}^i) : H_n^i \text{ is } \mathcal{F}_{n-1}\text{-measurable} \right\}.$$

For any $i = 0, \dots, l$, there exist $\widetilde{\omega}_{i0}, \dots, \widetilde{\omega}_{il(i)} \in \mathfrak{a}_{N-1}(\widetilde{\omega}_i)$ and $\widetilde{\alpha}_{i0}, \dots, \widetilde{\alpha}_{il(i)} \in \mathbb{R}_{++}$ such that $\widetilde{\omega}_{i0} = \widetilde{\omega}_i$, $\sum_{j=0}^{l(i)} \widetilde{\alpha}_{ij} = 1$, and

$$\sum_{j=0}^{l(i)} \widetilde{\alpha}_{ij} (S_N(\widetilde{\omega}_{ij}) - S_{N-1}(\widetilde{\omega}_{ij})) = 0.$$

Let $(i(0),j(0)),\dots,(i(m),j(m))$ be a numbering of the set $\{(i,j): i=0,\dots,l,\ j=0,\dots,l(i)\}$. We arrange this numbering in such a way that $i(0)=j(0)=0$. Set $\omega_k=\widetilde{\omega}_{i(k)j(k)}$, $\alpha_k=\widetilde{\alpha}_{i(k)}\widetilde{\alpha}_{i(k)j(k)}$, $k=0,\dots,m$. Then, for any

$$X=\sum_{n=1}^{N}\langle H_n,(S_n-S_{n-1})\rangle\in A,$$

we have

$$\begin{aligned}\sum_{k=0}^{m}\alpha_k X(\omega_k)&=\sum_{n=1}^{N-1}\sum_{i=0}^{l}\sum_{j=0}^{l(i)}\widetilde{\alpha}_i\widetilde{\alpha}_{ij}\langle H_n(\widetilde{\omega}_{ij}),(S_n(\widetilde{\omega}_{ij})-S_{n-1}(\widetilde{\omega}_{ij}))\rangle\\&\quad+\sum_{i=0}^{l}\sum_{j=0}^{l(i)}\widetilde{\alpha}_i\widetilde{\alpha}_{ij}\langle H_N(\widetilde{\omega}_{ij}),(S_N(\widetilde{\omega}_{ij})-S_{N-1}(\widetilde{\omega}_{ij}))\rangle\\&=\sum_{n=1}^{N-1}\sum_{i=0}^{l}\widetilde{\alpha}_i\langle H_n(\widetilde{\omega}_i),(S_n(\widetilde{\omega}_i)-S_{n-1}(\widetilde{\omega}_i))\rangle\\&\quad+\sum_{i=0}^{l}\widetilde{\alpha}_i\Big\langle H_N(\widetilde{\omega}_i),\sum_{j=0}^{l(j)}\widetilde{\alpha}_{ij}(S_N(\widetilde{\omega}_{ij})-S_{N-1}(\widetilde{\omega}_{ij}))\Big\rangle=0.\end{aligned}$$

In the second equality, we used the fact that $H_n$, $n=0,\dots,N$ and $S_n$, $n=0,\dots,N-1$ are constant on the atoms of $\mathcal{F}_{N-1}$. Thus, $\omega_1,\dots,\omega_m$ and $\alpha_0,\dots,\alpha_m$ satisfy the desired conditions. □

Proof of Theorem 4.1. *Step 1.* The implication (a)⇒(b) is obvious.

*Step 2.* Let us prove the implication (b)⇒(c). Suppose that there exist $m\in\{0,\dots,N-1\}$ and $\omega_0\in\Omega$ such that $S_m(\omega_0)\notin C_m^\circ(\omega_0)$. By the separation theorem, there exists $h\in\mathbb{R}^d$ such that $\langle h,(S_{m+1}(\omega)-S_m(\omega))\rangle\ge 0$ for any $\omega\in\mathfrak{a}_m(\omega_0)$ and $\langle h,(S_{m+1}(\omega)-S_m(\omega))\rangle>0$ for some $\omega\in\mathfrak{a}_m(\omega_0)$. Set

$$H_n(\omega)=\begin{cases}hI(\omega\in\mathfrak{a}_m(\omega_0)) & \text{if } n=m+1,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\sum_{n=1}^{N}\sum_{i=1}^{d}H_n^i(S_n^i-S_{n-1}^i)\in A\cap(L_+^0\setminus\{0\}),$$

which contradicts the NA condition.

*Step 3.* Let us prove the implication (c)⇒(d). Fix $D\in\mathcal{F}\setminus\{\emptyset\}$. Choose $\omega_0\in D$. Take $\omega_1,\dots,\omega_m\in\Omega$ and $\alpha_0,\dots,\alpha_m\in\mathbb{R}_{++}$ provided by Lemma 4.2. Then the measure $\mathsf{Q}=\sum_{k=0}^m\alpha_k\delta_{\omega_k}$ belongs to $\mathcal{M}$ and $\mathsf{Q}(D)>0$.

*Step 4.* Let us prove the implication (d)⇒(a). It has been shown in [3; Lem. 4.1] that $\mathcal{M}\subseteq\mathcal{R}$. Now it follows from Theorem 3.6 that the NGA is satisfied (note that the proof of the "if" part of that theorem does not employ Assumption 3.5). □

**Corollary 4.3.** *Suppose that* $S_0\in\mathbb{R}^d_{++}$,

$$\{(S_1(\omega),\dots,S_N(\omega)):\omega\in\Omega\}=(\mathbb{R}^d_{++})^N,$$

$\mathcal{F}_n=\mathcal{F}_n^S$, *and* $\mathcal{F}=\mathcal{F}_N$. *Then the model* $(\Omega,\mathcal{F},A)$ *satisfies the NGA condition.*

Proof. It is sufficient to note that, for any $\omega\in\Omega$ and $n=0,\dots,N-1$, we have $C_n^\circ(\omega)=\mathbb{R}^d_{++}$, so that condition (c) of Theorem 4.1 is satisfied. □

# 5 Continuous-Time Model with Finite Number of Assets

We will first consider the frictionless model (its probability version was discussed in [3; Sect. 4]). Thus, we are given a possibility space $(\Omega, \mathcal{F})$ endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and a family $(S_t)_{t\in[0,T]}$ of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable functions such that, for any $\omega$, the map $t \mapsto S_t(\omega)$ is càdlàg. We will assume that each component of $S$ is strictly positive (this condition is naturally satisfied if, for example, each $S^i$ is the price process of an equity or an option). The set of attainable incomes is defined by

$$A = \biggl\{\sum_{n=1}^{N}\sum_{i=1}^{d} H_n^i(S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \le \dots \le u_N \\ \text{are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable}\biggr\}.$$

It follows from the results of [3; Sect. 4] that if each component of $S$ is bounded below, then

$$\mathcal{R} = \mathcal{R}\biggl(\sum_{i=1}^{d}(S_T^i - S_0^i)\biggr) = \{\mathsf{Q} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\}.$$

We present two sufficient conditions for the absence of generalized arbitrage.

**Proposition 5.1.** *Suppose that* $S_0 \in \mathbb{R}_{++}^d$,

$$\{S_\cdot(\omega) : \omega \in \Omega\} = \{f : f \text{ is a càdlàg piecewise constant function } [0,T] \to \mathbb{R}_{++}^d \\ \text{with a finite number of jumps, } f(0) = S_0\},$$

$\mathcal{F}_t = \mathcal{F}_t^S$, *and* $\mathcal{F} = \mathcal{F}_T$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

Proof. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Let $0 < t_1 < \dots < t_N \le T$ be the jump times of $S_\cdot(\omega_0)$. We set $t_0 = 0$, $t_{N+1} = T$. Consider the sequence $\widetilde{S}_n = S_{t_n}$. For this sequence, the set $C_n^\circ(\omega)$ defined in the previous section equals $\mathbb{R}_{++}^d$ for any $\omega \in \Omega$ and $n = 0, \dots, N-1$. Thus, we can apply Lemma 4.2, which yields the existence of $\omega_1, \dots, \omega_m \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $S_\cdot(\omega_k)$ is constant on $[t_l, t_{l+1})$, $k = 0, \dots, m$, $l = 0, \dots, N$, $\sum_{k=0}^m \alpha_k = 1$, and the sequence $(S_{t_0}, \dots, S_{t_{N+1}})$ is an $(\mathcal{F}_{t_0}, \dots, \mathcal{F}_{t_{N+1}})$-martingale with respect to the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$. As $S$ is $\mathsf{Q}$-a.s. constant on $[t_l, t_{l+1})$, $l = 0, \dots, N$, the process $(S_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. This means that $\mathsf{Q} \in \mathcal{R}$. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 3.6 completes the proof. □

**Proposition 5.2.** *Suppose that* $S_0 \in \mathbb{R}_{++}^d$,

$$\{S_\cdot(\omega) : \omega \in \Omega\} = \bigl\{f : f \text{ is a càdlàg function } [0,T] \to \mathbb{R}_{++}^d \text{ with finite variation} \\ \text{such that, for any } i,\ \inf_{t\in[0,T]} f^i(t) > 0, \text{and } f(0) = S_0\bigr\}, \tag{5.1}$$

$\mathcal{F}_t = \mathcal{F}_t^S$, *and* $\mathcal{F} = \mathcal{F}_T$. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

**Proof.** Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Set $\varphi(t) = S_t(\omega_0)$, $\psi^i(t) = \ln \varphi^i(t)$, $i = 1, \dots, d$, $t \in [0,T]$. For each $i$, the function $\psi^i$ can be represented as $\psi^i = \psi^i_+ - \psi^i_-$, where $\psi^i_+$ and $\psi^i_-$ are càdlàg and increasing. Set

$$\lambda^i_+(t) = \frac{\psi^i_-(t)}{e-1}, \quad \lambda^i_-(t) = \frac{\psi^i_+(t)}{1-e^{-1}}, \quad i = 1, \dots, d, \ t \in [0,T].$$

Let $N^i_+$, $N^i_-$, $i = 1, \dots, d$ be independent Poisson processes with intensity 1. For each $i = 1, \dots, d$, the process

$$Z^i_t = \exp\bigl\{(N^i_+)_{\lambda^i_+(t)} - (N^i_-)_{\lambda^i_-(t)} - (e-1)\lambda^i_+(t) + (1-e^{-1})\lambda^i_-(t)\bigr\}, \quad t \in [0,T]$$

is a martingale with respect to its natural filtration. Let us denote the space of functions standing in (5.1) by $\mathcal{V}$. It is equipped with the $\sigma$-field $\mathcal{G} = \sigma(X_t; t \in [0,T])$, where $X_t(f) = f(t)$. Set $\mathsf{Q}_0 = \operatorname{Law}(Z_t; t \in [0,T])$. Then $X$ is an $(\mathcal{F}^X_t, \mathsf{Q}_0)$-martingale. In view of the representation

$$Z^i_t = \varphi^i(t) \exp\bigl\{(N^i_+)_{\lambda^i_+(t)} - (N^i_-)_{\lambda^i_-(t)}\bigr\}, \quad i = 1, \dots, d, \ t \in [0,T],$$

we have $\mathsf{Q}_0(\{\varphi\}) > 0$. Define the measure $\mathsf{Q}$ on $\bigl\{S_\cdot^{-1}(C) : C \in \mathcal{G}\bigr\}$ by $\mathsf{Q}(S_\cdot^{-1}(C)) := \mathsf{Q}_0(C)$. Note that $\{S_\cdot^{-1}(C) : C \in \mathcal{G}\} = \mathcal{F}$ and $\mathsf{Q}$ is correctly defined. Then $S$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. This means that $\mathsf{Q} \in \mathcal{R}$. Moreover, the set $S_\cdot^{-1}(\{\varphi\})$ contains $\omega_0$ and is an atom of $\mathcal{F}$. Hence, $S_\cdot^{-1}(\{\varphi\}) \subseteq D$, and therefore, $\mathsf{Q}(D) > 0$. An application of Theorem 3.6 completes the proof. □

The following statement is rather surprising.

**Proposition 5.3.** *Suppose that* $S_0 \in \mathbb{R}^d_{++}$,

$$\{S_\cdot(\omega) : \omega \in \Omega\} = \{f : f \text{ is a continuous function } [0,T] \to \mathbb{R}^d_{++},\ f(0) = S_0\},$$

$\mathcal{F}_t = \mathcal{F}^S_t$, *and* $\mathcal{F} = \mathcal{F}_T$. *Then the model* $(\Omega, \mathcal{F}, A)$ *does not satisfy the NGA condition.*

**Proof.** Suppose that the NGA condition is satisfied. By Theorem 3.6, there exists a measure $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(D) > 0$, where $D = \{S^1_t = 1 + t,\ t \in [0,T]\}$. Then $S$ should be an $(\mathcal{F}_t, \mathsf{Q})$-martingale (see [3; Sect. 4]). Moreover, $S$ is continuous. On the set $D$, the quadratic variation of $S^1$ is 0. This implies that $S^1_T = S^1_0$ $\mathsf{Q}$-a.e. on $D$ (see [8; Ch. IV, Prop. 1.13]). The obtained contradiction shows that the NGA condition is not satisfied. □

Let us now consider the model with proportional transaction costs (its probability version was discussed in [4; Sect. 3]). For this model, the set of discounted incomes is defined as

$$\begin{aligned} A = \Biggl\{ &\sum_{n=0}^{N} \sum_{i=1}^{d} \bigl[-H^i_n I(H^i_n > 0) S^i_{u_n} - H^i_n I(H^i_n < 0)(1-\lambda^i) S^i_{u_n}\bigr] : \\ &N \in \mathbb{N},\ u_0 \le \dots \le u_N \text{ are } (\mathcal{F}_t)\text{-stopping times}, \\ &H^i_n \text{ is } \mathcal{F}_{u_n}\text{-measurable, and } \sum_{n=0}^{N} H_n = 0 \Biggr\}. \end{aligned}$$

Here $\lambda^i \in [0,1)$ means the coefficient of proportional transaction costs for the $i$-th asset. For this model, we are able to prove the absence of generalized arbitrage under more natural assumptions than those used for the frictionless model.

**Proposition 5.4.** *Suppose that $S_0 \in \mathbb{R}^d_{++}$,*

$$\{S_\cdot(\omega) : \omega \in \Omega\} = \{f : f \text{ is a continuous function } [0,T] \to \mathbb{R}^d_{++},\ f(0) = S_0\},$$

*$\mathcal{F}_t = \mathcal{F}^S_t$, and $\mathcal{F} = \mathcal{F}_T$. Suppose moreover that $\lambda^i > 0$ for any $i$. Then the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition.*

Proof. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Consider the function $\varphi(t) = S_t(\omega_0)$. Fix $i \in \{1, \dots, d\}$ and set $\Delta^i = \inf_{t\in[0,T]} S^i_t(\omega_0)$. We can find points $0 = t_0 < \dots < t_M = T$ such that

$$|S^i_t(\omega_0) - S^i_{t_m}(\omega_0)| < \lambda^i \Delta^i / 3 \quad \text{for } m = 0, \dots, M-1,\ t \in [t_m, t_{m+1}).$$

Then the function $\psi^i$ defined as $(1-\lambda^i/2)\varphi^i(t_m)$ for $t \in [t_m, t_{m+1})$ is piecewise constant, $\psi^i(0) = S_0(\omega_0)$, and

$$(1 - \lambda^i)\varphi^i(t) \le \psi^i(t) \le \varphi^i(t), \quad i = 1, \dots, d,\ t \in [0,T]. \tag{5.2}$$

Set $\psi(t) = (\psi^1(t), \dots, \psi^d(t))$ and take $\omega'_0 \in \Omega$ such that $S_t(\omega'_0) = \psi(t)$. The reasoning used in the proof of Proposition 5.1 shows that there exist $\omega'_1, \dots, \omega'_m \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $\sum_{k=0}^m \alpha_k = 1$ and $S$ is a martingale with respect to the measure $\mathsf{Q}_0 = \sum_{k=0}^m \lambda_k \delta_{\omega'_k}$. Set $\omega_k = \omega'_k$, $k = 1, \dots, m$. Consider an arbitrary element

$$X = \sum_{n=0}^N \sum_{i=1}^d \bigl[-H^i_n I(H^i_n > 0) S^i_{u_n} - H^i_n I(H^i_n < 0)(1 - \lambda^i) S^i_{u_n}\bigr] \in A.$$

Set

$$Y = \sum_{n=0}^N \sum_{i=1}^d \bigl[-H^i_n I(H^i_n > 0) S^i_{u_n} - H^i_n I(H^i_n < 0) S^i_{u_n}\bigr].$$

In view of (5.2), $X(\omega_0) \le Y(\omega'_0)$, and hence,

$$\sum_{k=0}^m \alpha_k X(\omega_k) \le \alpha_0 Y(\omega'_0) + \sum_{k=1}^m \alpha_k X(\omega_k) \le \sum_{k=0}^m \alpha_k Y(\omega'_k) = \mathsf{E}_{\mathsf{Q}_0} Y.$$

Using the fact that $S$ is an $(\mathcal{F}_t, \mathsf{Q}_0)$-martingale and employing the representation

$$Y = \sum_{n=1}^N \sum_{i=1}^d \biggl(\sum_{k=1}^{n-1} H^i_k\biggr)(S^i_{u_n} - S^i_{u_{n-1}}),$$

we conclude that $\mathsf{E}_{\mathsf{Q}_0} Y = 0$ (note that $\mathsf{Q}_0$ is concentrated on a finite set of points). Thus, the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$ belongs to $\mathcal{R}$. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 3.6 completes the proof (Assumption 3.5 is satisfied in this model; see [3; Lem. 3.1]). □

# 6 Model with European Call Options as Basic Assets

We will consider a model with no transaction costs (its probability version was discussed in [3; Sect. 6]). Thus, we are given a possibility space $(\Omega, \mathcal{F})$ and $T \in [0, \infty)$. Let $S_T$

be an $\mathbb{R}_+$-valued $\mathcal{F}$-measurable function. From the financial point of view, $S_T$ is the price of some asset at time $T$. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices of European call options on this asset with maturity $T$ and let $\varphi(K)$, $K \in \mathbb{K}$ be the price at time 0 of a European call option with the payoff $(S_T - K)^+$. The set of attainable incomes is defined as

$$A = \left\{ \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) : N \in \mathbb{N},\ K_n \in \mathbb{K},\ h_n \in \mathbb{R} \right\}.$$

We assume that $0 \in \mathbb{K}$, which means the possibility to trade the underlying asset. Consider $Z_0 = S_T - \varphi(0)$. Then, for any $\mathsf{Q} \in \mathcal{R}(Z_0)$ and any $X \in A$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X = 0$. Thus, Assumption 3.5 is satisfied.

This model will be studied in two (most important) cases:

**1.** the case, where $\mathbb{K} = \mathbb{R}_+$;

**2.** the case, where $\mathbb{K}$ is finite.

Propositions 6.1 and 6.2 show that in case 1 the NGA condition is not satisfied in most natural situations, while in case 2 the NGA condition is satisfied in most natural situations.

Below $\varphi'_+$ denotes the right-hand derivative and $\varphi''$ denotes the second derivative of a convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ taken in the sense of distributions (i.e. $\varphi''((a,b]) = \varphi'_+(b) - \varphi'_+(a)$) with the convention: $\varphi''(\{0\}) = \varphi'_+(0) + 1$ (thus, $\varphi''$ is a probability measure provided that $\varphi'_+(0) \ge -1$).

**Proposition 6.1.** *Let $\mathbb{K} = \mathbb{R}_+$. Then the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition if and only if*

(a) *$\varphi$ is convex;*
(b) $\varphi'_+(0) \ge -1$;
(c) $\lim_{x\to\infty} \varphi(x) = 0$;
(d) *the set $C := \{S_T(\omega) : \omega \in \Omega\}$ is countable;*
(e) *$\varphi''$ is concentrated on $C$;*
(f) *$\varphi''(\{x\}) > 0$ for any $x \in C$.*

Proof. *Step 1.* Let us prove the "only if" implication. If the NGA is satisfied, then (by Theorem 3.6), for any $a \in C$, there exists a risk-neutral measure $\mathsf{Q}$ such that $\mathsf{Q}(S_T = a) > 0$. We have

$$\mathsf{E}_\mathsf{Q}(S_T - K)^+ = \varphi(K), \quad K \in \mathbb{R}_+,$$

which immediately implies (a)–(c). Furthermore, it follows that $\mathrm{Law}_\mathsf{Q}\, S_T = \varphi''$. In particular, $\varphi''(\{a\}) = \mathsf{Q}(S_T = a) > 0$, which yields (f), and (f) leads to (d). Employing once more the property $\mathrm{Law}_\mathsf{Q}\, S_T = \varphi''$, we get (e).

*Step 2.* Let us prove the "if" part. Let $a_1, a_2, \dots$ be a numbering of $C$. Find $\omega_1, \omega_2, \dots$ such that $S_T(\omega_i) = a_i$ and consider the measure $\mathsf{Q} = \sum_i \varphi''(\{a_i\})\delta_{\omega_i}$. Then

$$\mathrm{Law}_\mathsf{Q}\, S_T = \sum_i \varphi''(\{a_i\})\delta_{a_i} = \varphi''.$$

Hence,

$$\mathsf{E}_\mathsf{Q}(S_T - K)^+ = \int_{\mathbb{R}_+} (x - K)^+ \varphi''(dx) = \varphi(K), \quad K \in \mathbb{R}_+,$$

which means that $\mathsf{Q}$ is a risk-neutral measure. Furthermore, $\mathsf{Q}(S_T = a) = \varphi''(\{a\}) > 0$ for any $a \in C$. By Theorem 3.6, the NGA is satisfied. □

**Proposition 6.2.** *Suppose that $\mathbb{K}$ is finite, $0 \in \mathbb{K}$, and $\{S_T(\omega) : \omega \in \Omega\} = \mathbb{R}_{++}$. Then the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition if and only if*

(a) *$\varphi$ is strictly positive on $\mathbb{K}$;*
(b) *$\varphi$ is strictly convex on $\mathbb{K}$;*
(c) *$\varphi$ is strictly decreasing on $\mathbb{K}$;*
(d) *$\varphi(x) > \varphi(0) - x$, $x \in \mathbb{K} \setminus \{0\}$.*

Proof. *Step 1.* Let us prove the "only if" implication. If the NGA is satisfied, then, for any $a \in \mathbb{R}_{++}$, there exists a risk-neutral measure $\mathsf{Q}$ such that $\mathsf{Q}(S_T = a) > 0$. The function $\psi(x) := \mathsf{E}_\mathsf{Q}(S_T - x)^+$, $x \in \mathbb{R}_+$ is positive, convex, decreasing, $\psi(x) \geq \psi(0) - x$ for any $x \in \mathbb{R}_+$, $\psi'$ has a jump at the point $a$, and $\psi$ coincides with $\varphi$ on $K$. This yields (a)–(d).

*Step 2.* Let us prove the "if" part. Fix $a \in \mathbb{R}_{++}$. We can find a piecewise linear function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi$ is convex, $\psi'_+(0) = -1$, $\lim_{x\to\infty} \psi(x) = 0$, $\psi''(\{a\}) > 0$, and $\psi$ coincides with $\varphi$ on $K$. The measure $\psi''$ is concentrated on a countable set $\{a_1, a_2, \dots, \}$. Find $\omega_i$ such that $S_T(\omega_i) = a_i$ and consider the measure $\mathsf{Q} = \sum_i \varphi''(\{a_i\})\delta_{\omega_i}$. Then

$$\mathsf{E}_\mathsf{Q}(S_T - K)^+ = \psi(K) = \varphi(K), \quad K \in \mathbb{K},$$

which means that $\mathsf{Q}$ is a risk-neutral measure. Furthermore, $\mathsf{Q}(S_T = a) = \psi''(\{a\}) > 0$. By Theorem 3.6, the NGA condition is satisfied. □

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