GENERAL ARBITRAGE PRICING MODEL: PROBABILITY APPROACH

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Abstract. The purpose of this paper is to present a unified approach to pricing contingent claims through a new concept of *generalized arbitrage*.

First, we prove the fundamental theorem of asset pricing and establish the form of the fair price intervals within the framework of a *general arbitrage pricing model*.

Furthermore, these results are "projected" on several models, including:

- a dynamic model with an infinite number of assets;
- a model with European call options as basic assets;
- a mixed model.

This leads us, in particular, to the revision of the fundamental theorem of asset pricing for continuous-time models. Our variant of this theorem states that the absence of generalized arbitrage is equivalent to the existence of an equivalent measure, with respect to which the discounted price process is a true martingale. In a model with the infinite time horizon, uniformly integrable martingales come into play.

The general approach mentioned above allows us to narrow the fair price intervals by taking into consideration the current prices of traded derivatives.

Key words and phrases. Change of numéraire, general arbitrage pricing model, generalized arbitrage, fair price, fundamental theorem of asset pricing, martingale measure, martingale measure with given marginals, risk-neutral measure, set of attainable incomes.

1 Introduction

1. Purpose of the paper. In the fundamental work [22], Harrison and Kreps introduced a general model of pricing by arbitrage. Their paper formed the basis of the martingale approach to arbitrage pricing. However, there are some technical problems inherent in their model. The main one descends from the assumption that the so-called marketed contingent claims should belong to L^2 (the model proposed later by Kreps [32] enables one to relax this assumption to the L^p-integrability with $p \geq 1$). This restriction is not very natural as shown by the example below.

Consider the following simple model for an asset's (discounted) price evolution: $S_0 = 1$, $S_1 = \xi$, $S_2 = \xi\eta$, where ξ and η are independent random variables, each taking on values 1/2 and 3/2 with probability 1/2 (S_n means the discounted price of some asset at time n). Let $(\mathcal{F}_n)_{n=0,1,2}$ be a filtration such that $\mathcal{F}_0$ is trivial, S is an $(\mathcal{F}_n)$-martingale, and $\mathcal{F}_1$ is rich enough, so that there exists an $\mathcal{F}_1$-measurable random variable H that is not integrable. Then $H(S_2-S_1)$ is a natural candidate for a marketed contingent claim. However, it does not belong to L^1.

Further development of arbitrage pricing theory was mainly concentrated on dynamic models with a finite number of assets, which may be viewed as particular cases of the model proposed by Harrison and Kreps. Harrison and Pliska [23] introduced the admissibility condition on the trading strategies as a substitute for the integrability restriction described above. The fundamental theorem of asset pricing (FTAP) for a discrete-time model was established in the papers [23] and [11] (alternative proofs were given in [26], [29], [30], [35], [37], and [41]). The FTAP for a continuous-time model was established in the papers [12] and [15] (another proof was given in [28]). In a series of papers [15], [18], [19], and [31], the form of upper and lower prices of a contingent claim in a continuous-time model was established. However, there are some serious problems inherent in the mentioned approach to continuous-time models (these problems are described in Examples 4.3, 4.4, 4.5, and especially in Example 4.6).

In this paper, we propose a *general arbitrage pricing model* that has the same spirit as the model of Harrison and Kreps, but avoids the problems described above. This approach allows us to consider in a simple and unified manner various models of arbitrage pricing theory, some of which have so far been investigated separately and by different techniques. These include

- static as well as dynamic models; (see Sections 4, 5);
- models with an infinite number of assets (in particular, this allows us to consider models with traded derivatives as basic assets, which makes it possible to narrow considerably fair price intervals — see Section 6);
- models with transaction costs (these are considered in the paper [5], which is a continuation of this paper);
- combinations of various models (see Section 7).

In the paper [5], we extend our results to models with transaction costs. Our approach to these models turns out to be different from the existing ones. Furthermore, in the paper [6], we introduce the *possibility approach* to arbitrage pricing, which enables one to get rid of such a vague object as the original probability measure.

2. General arbitrage pricing model. A general arbitrage pricing model is a quadruple $(\Omega,\mathcal{F},\mathsf{P},A)$, where $(\Omega,\mathcal{F},\mathsf{P})$ is a probability space and A (it is called the *set of attainable incomes*) is a collection of random variables on $(\Omega,\mathcal{F},\mathsf{P})$ meaning the set of discounted incomes one can obtain by trading certain assets. For a model $(\Omega,\mathcal{F},\mathsf{P},A)$, we introduce a notion of the *No Generalized Arbitrage* (NGA). The NGA condition might be viewed as a strengthening of the No Free Lunch condition known in financial mathematics (the necessity to strengthen the latter one is illustrated by Example 6.4). Furthermore, we define an equivalent *risk-neutral* measure as a measure $\mathsf{Q}\sim\mathsf{P}$ such that $\mathsf{E}_\mathsf{Q}X^-\ge\mathsf{E}_\mathsf{Q}X^+$ for any $X\in A$ (X^- and X^+ denote the negative part and the positive part of X, respectively; the expectations $\mathsf{E}_\mathsf{Q}X^-$, $\mathsf{E}_\mathsf{Q}X^+$ here are allowed to take on the value $+\infty$). Although the concept of a risk-neutral measure is a classical concept of financial mathematics, this particular definition seems to be new. It turns out to be very convenient as illustrated by considerations of Sections 4–7.

The first basic result of the paper is Theorem 3.6, which may be called the FTAP for the general arbitrage pricing model. It states (under some assumption that is automatically satisfied in the particular models considered below) that a model satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.

Next we consider the problem of pricing contingent claims. We define a fair price of a contingent claim F (F is a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$) as a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGA condition. The second basic result of the paper is Theorem 3.10. It states (under some natural assumptions) that the set of fair prices of F coincides with the interval $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\}$, where $\mathcal{R}$ denotes the set of equivalent risk-neutral measures.

3. Particular models. Various models of arbitrage pricing can be viewed as particular cases of the general model described above. In order to embed a particular model into this general framework, one should

1. specify the set A of attainable incomes;
2. find out the structure of the set of equivalent risk-neutral measures (typically, the risk-neutral measures in a particular model admit a simpler description than the general definition of a risk-neutral measure).

Once this is done, Theorem 3.6 gives the necessary and sufficient conditions for the absence of generalized arbitrage, while Theorem 3.10 yields the form of the set of fair prices of a contingent claim.

When "projected" to a discrete-time model with a finite number of assets, our results agree with the classical ones. Namely, the class of risk-neutral measures coincides with the class of martingale measures, while our intervals of fair prices coincide with the classical No Arbitrage intervals.

However, for continuous-time models (considered in Sections 4, 5), our results differ from the traditional ones. First of all, it should be mentioned that, unlike discrete-time models, the continuous-time models do not possess a unique universally accepted approach to pricing by arbitrage. The two most well-known approaches are: the "L^2-approach" proposed by Harrison and Kreps [22] and the approach developed in a series of papers [12], [15], [18], [19], [28], [31], and others. Our approach is different from the "L^2-approach" because we never impose any integrability restrictions on price processes or trading strategies.

Let us now describe the differences between our approach and the second one mentioned above. First, we consider the model with an arbitrary number of assets, while the traditional approach deals with a finite number of assets. Second, we consider only simple (i.e. piecewise constant) trading strategies with no admissibility condition imposed. Third, our FTAP states that a model with a finite time horizon satisfies the NGA condition if and only if there exists an equivalent measure, with respect to which the discounted price process is a true martingale; a model with the infinite time horizon satisfies the NGA condition if and only if there exists an equivalent measure, with respect to which the discounted price process is a uniformly integrable martingale. This is different from the traditional FTAP provided by Delbaen and Schachermayer [12], [15] (another proof was given by Kabanov [28]), which states that a model satisfies the No Free Lunch with Vanishing Risk (NFLVR) condition (defined through the general predictable admissible strategies) if and only if there exists an equivalent measure, with respect to which the discounted price process is a sigma-martingale (this class of processes has been introduced by Chou [8] and Émery [17]). Let us also point out in this connection that for the continuous-time model with a finite number assets, Sin [40] and

Yan [43] introduced some strengthenings of the NFLVR condition and proved that these strengthenings are equivalent to the existence of an equivalent measure, with respect to which the discounted price process is a true martingale. Thus, our FTAP agrees with these results although our NGA condition is different from the variants of No Arbitrage in these papers. Fourth, our definition of the interval of fair prices differs from the traditional one. We discuss in Section 4 the problems of the traditional theory of arbitrage pricing that arise when one considers admissible strategies, sigma-martingale measures, and traditional intervals of fair prices. These problems do not arise in our framework. Furthermore, it turns out that, unlike the NFLVR property, the NGA property is preserved under a change of numéraire (see Theorem 4.8).

The intervals of fair prices provided by arbitrage considerations are known to be unacceptably large in incomplete models. There are several ways to overcome this problem proposed by financial mathematics. One of them is to consider traded derivatives as basic assets. Typically, this leads to models with an infinite number of assets, and this often creates serious theoretical problems. Our approach can easily be applied to models with an infinite number of assets, and the traded derivatives can be taken into consideration as follows. The set A depends on the amount of traded securities that we take into account; the set $\mathcal{R}$ depends on A; the interval of fair prices depends on $\mathcal{R}$. Diagrammatically,

$$\text{Assets} \longrightarrow A \longrightarrow \mathcal{R} \longrightarrow \text{Interval of fair prices.}$$

When the amount of assets taken into consideration is enlarged (i.e. more prices of traded contracts are taken into account), the set A is enlarged, the set $\mathcal{R}$ is reduced, and the sets of fair prices are reduced.

In Section 6, we consider a model, which takes into account traded European call options on a fixed asset with a fixed maturity T. It is shown that if options with all positive strike prices are traded (of course, this is an idealized assumption, but it is typical for the theory), then the risk-neutral measure is unique. As a corollary, the fair price of a contingent claim depending only on the asset's price at time T (such are, for example, binary options) is uniquely determined.

It should be mentioned that this model was first proposed by Breeden and Litzenberger [2] and is very popular in mathematical finance (a literature review on this model is given in [25]). Our approach to this model is different from the existing ones. In particular, we establish the form of fair price intervals based on the NGA considerations, while traditionally the fair price of a contingent claim in this model is derived by representing the payoff as a combination of (a continuum of) European call options. This trick requires the smoothness of the payoff function (for instance, binary options do not satisfy this condition), while in our approach no smoothness or continuity requirements are imposed.

The general approach introduced in Section 3 admits an easy procedure of the combination of models. The aim of this procedure is to narrow the sets of fair prices by taking into consideration the current prices of a larger amount of traded contracts. Thus, the models of Section 4–6 may be viewed as "building blocks" for constructing mixed models. An example is provided by Section 7, in which we consider a mixed static-dynamic model. The "building blocks" are provided by the models of Sections 4 and 6. We show that, for the mixed model, the set $\mathcal{R}$ consists of the equivalent *martingale measures with given marginals*, i.e. the measures, with respect to which the discounted price process is a martingale with preassigned marginal distributions. Such measures have recently attracted attention in the literature (see [3], [4; Sect. 4.1], and [33]).

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2 Ordinary Arbitrage

In this section, we briefly describe the classical arbitrage pricing theory in a static model with a finite number of assets. This material is well-known (for more details, one may consult, for instance, [20; Ch. 1]).

The general arbitrage pricing model introduced in Section 3 may be regarded as the infinite-dimensional version of the model of this section (with the definitions of arbitrage and the definitions of fair prices appropriately reformulated).

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $S_0 \in \mathbb{R}^d$ and S_1 be an $\mathbb{R}^d$-valued random vector on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, S_n^i is the discounted price of the i-th asset at time n.

Consider the set

$$A = \left\{ \sum_{i=1}^{d} h^i (S_1^i - S_0^i) : h^i \in \mathbb{R} \right\}. \tag{2.1}$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading assets $1, \dots, d$ at times 0, 1.

Definition 2.1. A model $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$ satisfies the *No Arbitrage* (NA) condition if $A \cap L_+^0 = \{0\}$ (L_+^0 denotes the set of $\mathbb{R}_+$-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$).

Definition 2.2. An equivalent *martingale measure* is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E_Q}|S_1| < \infty$ and $\mathsf{E_Q} S_1 = S_0$. The set of equivalent martingale measures will be denoted by $\mathcal{M}$.

`Notation`. Set $C = \overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_{\mathsf{P}} S_1$, where "$\overline{\operatorname{conv}}$" denotes the closed convex hull, "supp" denotes the support, and $\operatorname{Law}_{\mathsf{P}} S_1$ is the distribution of S_1 under P. Let C° denote the relative interior of C, i.e. the interior of C in the relative topology of the smallest affine subspace of $\mathbb{R}^d$ containing C.

Theorem 2.3 (FTAP). *For the model $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$, the following conditions are equivalent:*

(a) *NA*;
(b) $S_0 \in C^\circ$;
(c) $\mathcal{M} \neq \emptyset$.

Proof. *Step 1.* Let us prove the implication (a)$\Rightarrow$(b). If $S_0 \notin C^\circ$, then, by the separation theorem, there exists a vector $h \in \mathbb{R}^d$ such that $\langle h, (x - S_0)\rangle \ge 0$ for any $x \in C$ and $\langle h, (x - S_0)\rangle > 0$ for some $x \in C$. This means that $\langle h, (S_1 - S_0)\rangle \ge 0$ P-a.s. and $\mathsf{P}(\langle h, (S_1 - S_0)\rangle > 0) > 0$. But this contradicts the NA condition.

Step 2. Let us prove the implication (b)$\Rightarrow$(c). The set

$$E = \{\mathsf{E}_\mathsf{Q} S_1 : \mathsf{Q} \sim \mathsf{P},\ \mathsf{E}_\mathsf{Q}|S_1| < \infty\}$$

is convex, and the closure of E contains $\operatorname{supp}\operatorname{Law}_\mathsf{P} S_1$. Consequently, $E \supseteq C^\circ$.

Step 3. Let us prove the implication (c)$\Rightarrow$(a). Take $\mathsf{Q} \in \mathcal{M}$. Then $\mathsf{E}_\mathsf{Q} X = 0$ for any $X \in A$. This implies the NA condition. □

Now, let F be a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, F is the discounted payoff of some contingent claim.

Definition 2.4. A real number x is a *fair price* of F if the model with $d+1$ assets $(\Omega, \mathcal{F}, \mathsf{P}, x, S_0^1, \dots, S_0^d, F, S_1^1, \dots, S_1^d)$ satisfies the NA condition. The set of fair prices of F will be denoted by $I(F)$.

Notation. Set $D = \overline{\operatorname{conv}} \operatorname{supp}\operatorname{Law}_\mathsf{P}(F, S_1)$ and let D° denote the relative interior of D.

Theorem 2.5 (Pricing contingent claims). *Suppose that the model* $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$ *satisfies the NA condition. Then*

$$I(F) = \{x : (x, S_0) \in D^\circ\} = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{M}\}.$$

The expectation $\mathsf{E}_\mathsf{Q} F$ *here is taken in the sense of finite expectations, i.e. we consider only those* Q, *for which* $\mathsf{E}_\mathsf{Q}|F| < \infty$.

This is a direct consequence of Theorem 2.3. □

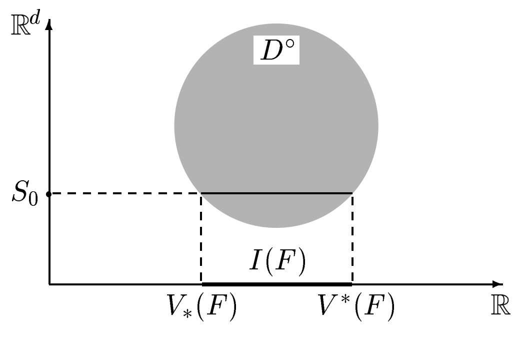

Figure 1. The joint arrangement of $I(F)$, $V_*(F)$, $V^*(F)$, and D°

Remark. Another way to define the fair price interval (which is commonly used in financial mathematics) is as follows. We introduce the lower and the upper prices by

$$V_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \text{ P-a.s.}\},$$
$$V^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \text{ P-a.s.}\},$$

and the fair price interval is defined as the interval with the endpoints $V_*(F)$ and $V^*(F)$ (to be more precise, if $V_*(F) < V^*(F)$, we consider the interval $(V_*(F), V^*(F))$; if $V_*(F) = V^*(F)$, we consider the one-point interval $\{V_*(F)\}$). One can easily check that if the model $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$ satisfies the NA condition, then the interval of fair prices defined this way coincides with the interval $I(F)$ introduced above (a proof can be found in [20; Th. 1.23]).

3 Generalized Arbitrage

Definition 3.1. A *general arbitrage pricing model* is a quadruple $(\Omega, \mathcal{F}, \mathsf{P}, A)$, where $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space and A is a convex cone in L^0 (L^0 is the space of real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ considered up to indistinguishability). The set A will be called the *set of attainable incomes.*

From the financial point of view, A is the set of discounted incomes that can be obtained by trading a certain amount of assets. An example is provided by (2.1). In the frictionless models, A is a linear space. In the models with transaction costs, A is a cone.

`Notation`. (i) Set

$$\begin{aligned} B = \Big\{ Z \in L^0 :\ & \text{there exist } (X_n)_{n\in\mathbb{N}} \in A \text{ and } a \in \mathbb{R} \\ & \text{such that } X_n \ge a \text{ } \mathsf{P}\text{-a.s. and } Z = \lim_{n\to\infty} X_n \text{ } \mathsf{P}\text{-a.s.} \Big\}. \end{aligned} \tag{3.1}$$

The elements of B might be regarded as generalized attainable incomes bounded below.

(ii) For $Z \in B$, denote $\gamma(Z) = 1 - \operatorname{essinf}_{\omega\in\Omega} Z(\omega)$ and set

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L^0_+\}, \\ A_2(Z) &= \left\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \right\}, \\ A_3(Z) &= A_2(Z) \cap L^\infty, \\ A_4(Z) &= \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, L^1(\mathsf{P})). \end{aligned} \tag{3.2}$$

Here L^0_+ is the set of $\mathbb{R}_+$-valued elements of L^0; L^∞ is the space of bounded elements of L^0; $\sigma(L^\infty, L^1(\mathsf{P}))$ denotes the weak topology on L^∞ induced by the space $L^1(\mathsf{P})$ of the P-integrable random variables on $(\Omega, \mathcal{F}, \mathsf{P})$.

Definition 3.2. A model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the *No Generalized Arbitrage* (NGA) condition if for any $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.

`Remarks`. (i) Note that $A_4(Z) \cap L^0_+ = \{0\}$ if and only if $A_5(Z) \cap L^0_+ = \{0\}$, where

$$A_5(Z) = \{(Z + \gamma(Z))X : X \in A_4(Z)\}. \tag{3.3}$$

The elements of $A_5(Z)$ might be regarded as generalized attainable incomes (i.e. one can approximate the elements of $A_5(Z)$ by the elements of A_1).

(ii) The existence of a generalized arbitrage opportunity means that there exist $Z \in B$, $W \in L^0_+ \setminus \{0\}$ and generalized sequences $(X_\lambda)_{\lambda\in\Lambda} \in A$, $(Y_\lambda)_{\lambda\in\Lambda} \in L^0_+$, and $(\alpha_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}_+$ such that $|X_\lambda - Y_\lambda| \le \alpha_\lambda(Z + \gamma(Z))$, $\lambda \in \Lambda$ and $(X_\lambda - Y_\lambda)$ converges to W in the sense that $\mathsf{E}_\mathsf{Q}(X_\lambda - Y_\lambda) \longrightarrow \mathsf{E}_\mathsf{Q} W$ for any probability measure $\mathsf{Q} \ll \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} Z < \infty$.

(iii) The NGA condition is similar to the *No Free Lunch* (NFL) condition introduced by Kreps [32] in a different framework. The NFL condition can be defined in our framework as: $A_4(0) \cap L^0_+ = \{0\}$. One can also define the *No Arbitrage* (NA) condition in our framework as: $A \cap L^0_+ = \{0\}$. The NGA condition is the strongest one: NGA$\Rightarrow$NFL, NGA$\Rightarrow$NA.

Definition 3.3. An equivalent *risk-neutral measure* is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X^- \ge \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$ (we use the notation $X^- = (-X) \vee 0$, $X^+ = X \vee 0$). The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of equivalent risk-neutral measures will be denoted by $\mathcal{R}$.

`Notation`. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of the probability measures $\mathsf{Q} \sim \mathsf{P}$ with the property: for any $X \in A$ such that $X \ge -\alpha Z - \beta$ P-a.s. with some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \le 0$.

Lemma 3.4. *For any $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

`Proof`. Take $Z \in B$, $\mathsf{Q} \in \mathcal{R}$. It follows from the Fatou lemma that Z is Q-integrable. Thus, if $X \in A$ satisfies the inequality $X \ge -\alpha Z - \beta$ P-a.s with some $\alpha, \beta \in \mathbb{R}_+$, then $\mathsf{E}_\mathsf{Q} X^- < \infty$. By the definition of $\mathcal{R}$, $\mathsf{E}_\mathsf{Q} X^+ \le \mathsf{E}_\mathsf{Q} X^-$. As a result, $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \le 0$. □

The following basic assumption is satisfied in all the particular models considered below.

Assumption 3.5. There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (in particular, both sets might be empty).

Theorem 3.6 (FTAP). *Suppose that Assumption 3.5 is satisfied. Then the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.*

The proof is based on a well-known result of Kreps [32] and Yan [42] (its proof can also be found in [37], [41], and other papers):

Lemma 3.7 (Kreps, Yan). *Let C be a $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone in L^∞ such that $C \supseteq L^\infty_-$ (L^∞_- is the set of negative elements of L^∞) and $C \cap L^0_+ = \{0\}$. Then there exists a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X \le 0$ for any $X \in C$.*

`Proof of Theorem 3.6`. *Step 1.* Let us prove the "only if" implication. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 3.7 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A_4(Z_0)$ yields a probability measure $\mathsf{Q}_0 \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \le 0$ for any $X \in A_4(Z_0)$. By the Fatou lemma, for any $X \in A$ such that $\frac{X}{Z_0 + \gamma(Z_0)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0 + \gamma(Z_0)} \le 0$ (note that $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0 + \gamma(Z_0)} \wedge c \le 0$ for any $c > 0$). Consider the probability measure $\mathsf{Q} = \frac{c}{Z_0 + \gamma(Z_0)} \mathsf{Q}_0$, where c is the normalizing constant (it exists since $Z_0 + \gamma(Z_0) \ge 1$). Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$.

Step 2. Let us prove the "if" implication. Take $\mathsf{Q} \in \mathcal{R}$ and $Z \in B$. It follows from the Fatou lemma that Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is the normalizing constant. For any $X \in A$ such that $\frac{X}{Z+\gamma(Z)}$ is bounded below by a constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_{\mathsf{Q}} X^- \le \mathsf{E}_{\mathsf{Q}}(\alpha Z + \alpha\gamma(Z)) < \infty,$$

and consequently,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{X}{Z+\gamma(Z)} = c\mathsf{E}_{\mathsf{Q}} X \le 0.$$

Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} X \le 0$ for any $X \in A_4(Z)$. As a result, $A_4(Z) \cap L^0_+ = \{0\}$. □

It is seen from the above proof that the implication $\mathcal{R} \neq \emptyset \Rightarrow$ NGA is true without Assumption 3.5. The following example shows that this assumption is essential for the reverse implication.

Example 3.8. Let $(X_t)_{t\in[0,1]}$ be a collection of independent Gaussian random variables with mean 1 and variance 1 defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $\mathcal{F} = \sigma(X_t; t \in [0,1])$ and

$$A = \left\{ \sum_{n=1}^{N} h_n X_{t_n} : N \in \mathbb{N},\ t_n \in [0,1],\ h_n \in \mathbb{R} \right\}.$$

Clearly, the only element of A that is bounded below is 0. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition.

Suppose now that there exists an equivalent risk-neutral measure Q. Set $\rho = \frac{d\mathsf{Q}}{d\mathsf{P}}$. Note that $\mathcal{F} = \cup_C \sigma(X_t; t \in C)$, where the union is taken over all the countable sets $C \subset [0,1]$. Hence, there exists a countable set $C_0 \subset [0,1]$ such that ρ is $\sigma(X_t; t \in C_0)$-measurable. For any $t \notin C_0$, we have

$$\mathsf{E}_{\mathsf{Q}} X_t = \mathsf{E}_{\mathsf{P}} \rho X_t = \mathsf{E}_{\mathsf{P}} \rho \cdot \mathsf{E}_{\mathsf{P}} X_t = \mathsf{E}_{\mathsf{P}} X_t = 1.$$

As a result, there exists no equivalent risk-neutral measure. □

Now, let F be a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ meaning the discounted payoff of a contingent claim.

Definition 3.9. A real number x is a *fair price* of F if the extended model $(\Omega, \mathcal{F}, \mathsf{P}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGA condition. (From the financial point of view, $A + \{h(F-x) : h \in \mathbb{R}\}$ is the set of discounted incomes that can be obtained by trading the "original" assets as well as trading the contract F at the price x.) The set of fair prices of F will be denoted by $I(F)$.

Theorem 3.10 (Pricing contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies Assumption 3.5 and the NGA condition, while F is bounded below. Then*

$$I(F) = \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\}.$$

The expectation $\mathsf{E}_{\mathsf{Q}} F$ here is taken in the sense of finite expectations, i.e. we consider only those Q, for which $\mathsf{E}_{\mathsf{Q}} F < \infty$.

Proof. *Step 1.* Let $x \in I(F)$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Set $Z_1 = Z_0 + (F - x)$. Then $Z_1 \in B'$, where B' is defined by (3.1) with A replaced by $A' := A + \{h(F-x) : h \in \mathbb{R}\}$. Lemma 3.7 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A_4'(Z_1)$ ($A_4'(Z_1)$ is defined by (3.2)) yields a probability measure $\mathsf{Q}_0 \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \le 0$ for any $X \in A_4'(Z_1)$. By the Fatou lemma, for any $X \in A'$ such that $\frac{X}{Z_1 + \gamma(Z_1)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_1+\gamma(Z_1)} \le 0$. Consider the probability measure $\mathsf{Q} = \frac{c}{Z_1+\gamma(Z_1)}\mathsf{Q}_0$, where c is the normalizing constant (it exists since $Z_1 + \gamma(Z_1) \ge 1$). Then $\mathsf{Q} \in \mathcal{R}(Z_1) \subseteq \mathcal{R}(Z_0) = \mathcal{R}$. Moreover, $\mathsf{E}_\mathsf{Q}(x - F) \le 0$ and $\mathsf{E}_\mathsf{Q}(F - x) \le 0$ since the random variables $\frac{x-F}{Z_1+\gamma(Z_1)}$ and $\frac{F-x}{Z_1+\gamma(Z_1)}$ are bounded below. Thus, $x = \mathsf{E}_\mathsf{Q} F$.

Step 2. Now, let $x = \mathsf{E}_\mathsf{Q} F$, where $\mathsf{Q} \in \mathcal{R}$. Take $Z \in B'$. Choose an arbitrary element $Y = X + h(F-x) \in A'$ (here $X \in A$) such that Y is bounded below. It follows from the condition $x = \mathsf{E}_\mathsf{Q} F$ that $\mathsf{E}_\mathsf{Q} X^- < \infty$. As $\mathsf{Q} \in \mathcal{R}$, we have $\mathsf{E}_\mathsf{Q} X \le 0$. This implies that $\mathsf{E}_\mathsf{Q} Y \le 0$. By the Fatou lemma, Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is the normalizing constant. For any $Y = X + h(F-x) \in A'$ such that $\frac{Y}{Z+\gamma(Z)}$ is bounded below by some constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_\mathsf{Q} Y^- \le \mathsf{E}_\mathsf{Q}(\alpha Z + \alpha\gamma(Z)) < \infty.$$

Consequently, $\mathsf{E}_\mathsf{Q} X^- < \infty$, $\mathsf{E}_\mathsf{Q} X \le 0$, and $\mathsf{E}_\mathsf{Q} Y \le 0$. This means that $\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{Y}{Z+\gamma(Z)} \le 0$. Hence, for any $Y \in A_4'(Z)$, we have $\mathsf{E}_{\widetilde{\mathsf{Q}}} Y \le 0$. This implies that $A_4'(Z) \cap L_+^0 = \{0\}$. As a result, $x \in I(F)$. □

Remarks. (i) Theorem 3.10 remains valid if the condition "F is bounded below" is replaced by the condition "F is bounded above" (the proof remains the same).

(ii) Another way to define the fair price intervals could be as follows. We introduce the lower and the upper prices by

$$V_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \ \mathsf{P}\text{-a.s.}\}, \tag{3.4}$$

$$V^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \ \mathsf{P}\text{-a.s.}\}, \tag{3.5}$$

and the fair price interval is defined as the interval with the endpoints $V_*(F)$ and $V^*(F)$. However, unlike the model of Section 2, in a general model the interval defined this way might be larger than $I(F)$ (see Example 6.5).

To conclude the section, we "project" our results on the model of Section 2.

Example 3.11. Consider the model of Section 2 and assume additionally that the components of S_1 are bounded below. Then, clearly, the class of risk-neutral measures coincides with the class of martingale measures. Consequently, the NGA turns out to be equivalent to the NA and the fair price interval based on the NGA coincides with the fair price interval based on the NA.

4 Dynamic Model with Finite Time Horizon

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t^i)_{t\in[0,T]}$, $i \in I$ be a family of real-valued $(\mathcal{F}_t)$-adapted càdlàg processes. Here, I is an arbitrary set (it might be finite or infinite). From the financial point of view, S_t^i is the discounted price of the i-th asset at time t. Define the set of attainable incomes by

$$A = \Big\{ \sum_{n=1}^{N} \sum_{i \in I} H_n^i (S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \le \dots \le u_N \text{ are } (\mathcal{F}_t)\text{-stopping times},$$

$$H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H_n^i = 0 \text{ for all } i, \text{ except for a finite set} \Big\}. \tag{4.1}$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading assets from I on the interval $[0,T]$.

We will assume that each process S^i is bounded below (most financial assets automatically satisfy this condition). Moreover, we assume that there exists $Z_0 \in B$ (B is defined by (3.1)) with the property: for any $i \in I$, there exist $\alpha, \beta > 0$ such that $S_T^i \le \alpha Z_0 + \beta$ a.s. This assumption is automatically satisfied in natural models.

Indeed, if I is finite, then the above assumption is satisfied with

$$Z_0 = \sum_{i \in I} (S_T^i - S_0^i).$$

If I is countable, then the above assumption is satisfied with

$$Z_0 = \sum_{i \in I} \lambda^i (S_T^i - S_0^i),$$

where constants $\lambda^i > 0$ are chosen in such a way that $\sum_{i \in I} \lambda^i S_T^i < \infty$ a.s. and $\sum_{i \in I} \lambda^i S_0^i < \infty$.

If S is the discounted price process of some asset and S^i is the discounted price process of a European call option on this asset with maturity T and strike price i, then $S_T^i = (S_T - i)^+$, and hence, the above assumption is satisfied with $Z_0 = S_T - S_0$ (we assume that the process S is included in the collection $(S^i)_{i \in I}$).

If S^i is the discounted price process of a zero-coupon bond with maturity i, then S^i takes on values in $[0,1]$, and the above assumption is satisfied with $Z_0 = 0$.

In order to get the FTAP and to obtain the form of the fair price intervals, it is sufficient to prove that Assumption 3.5 is satisfied and to find the structure of risk-neutral measures. We call the corresponding statement the *Key Lemma* of the section.

`Notation`. Set $\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for any } i \in I,\ S^i \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\}$.

Key Lemma 4.1. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}(Z_0) = \mathcal{M}.$$

The proof employs the following statement (see [26] or [38; Ch. II, § 1c]):

Lemma 4.2. *Let* $(X_n)_{n=0,\dots,N}$ *be an* $(\mathcal{F}_n)$*-local martingale such that* $\mathsf{E}|X_0| < \infty$ *and* $\mathsf{E}X_N^- < \infty$. *Then* X *is an* $(\mathcal{F}_n)$*-martingale.*

`Proof of Key Lemma 4.1.` *Step 1.* The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 3.4.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. Fix $i \in I$. For any $u \in [0,T]$, the random variable $S_u^i - S_0^i$ is bounded below, and therefore, $\mathsf{E}_{\mathsf{Q}}(S_u^i - S_0^i) \le 0$.

In particular, S_u^i is Q-integrable. For any $u \le v \in [0,T]$ and any $D \in \mathcal{F}_u$ such that S_u^i is bounded on D, the random variable $I_D(S_v^i - S_u^i)$ is bounded below, and hence, $\mathsf{E}_\mathsf{Q} I_D(S_v^i - S_u^i) \le 0$. This proves that S^i is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale. It follows from the assumption $S_T^i \le \alpha Z_0 + \beta$ and the definition of $\mathcal{R}(Z_0)$ that $\mathsf{E}_\mathsf{Q}(S_T^i - S_0^i) = 0$. This implies that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix

$$X = \sum_{n=1}^{N} \sum_{i \in I} H_n^i (S_{u_n}^i - S_{u_{n-1}}^i).$$

The process

$$M_n = \sum_{k=1}^{n} \sum_{i \in I} H_k^i (S_{u_k}^i - S_{u_{k-1}}^i), \quad n = 0, \dots, N$$

is a Q-local martingale with respect to the filtration $(\mathcal{F}_{u_k})$. Now, it follows from Lemma 4.2 that $\mathsf{E}_\mathsf{Q} X^- \ge \mathsf{E}_\mathsf{Q} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. $\square$

Remark. If the NGA condition is satisfied, then each S^i is an $(\mathcal{F}_t, \mathsf{P})$-semimartingale. This follows from the fact that the semimartingale property is preserved under an equivalent change of measure (see [27; Ch. III, Th. 3.13]).

For discrete-time models with a finite number of assets the approach proposed here agrees with the classical one: the NGA condition is equivalent to the existence of an equivalent martingale measure, which, in turn, is equivalent to the NA condition; the interval of fair prices of a contingent claim that is bounded below coincides with the classical one. However, for continuous-time models with a finite number of assets our approach turns out to be completely different from the traditional approach developed in [12], [15], [18], [19], [28], [31]. Let us briefly describe the latter one.

In the traditional approach, the discounted price process S is assumed to be an $\mathbb{R}^d$-valued $(\mathcal{F}_t, \mathsf{P})$-semimartingale. The "set of attainable incomes" (although this term is not used in the traditional approach) has the form

$$\begin{aligned} A = \Big\{ \int_0^T H_u dS_u : \ & H \text{ is an } \mathbb{R}^d\text{-valued } (\mathcal{F}_t)\text{-predictable } S\text{-integrable} \\ & \text{process satisfying the } \textit{admissibility} \text{ condition, i.e. there exists} \\ & a \in \mathbb{R} \text{ such that } \int_0^t H_u dS_u \ge a \text{ for any } t \in [0,T] \Big\}. \end{aligned} \tag{4.2}$$

(Here $\int_0^t H_u dS_u$ is the vector stochastic integral; its definition can be found in [27; Ch. III, § 6c] or [39]). Consider the sets

$$\begin{aligned} &A_1 = \{X - Y : X \in A,\ Y \in L_+^0\}, \\ &A_2 = A_1 \cap L^\infty, \\ &A_3 = \text{closure of } A_2 \text{ in the norm topology of } L^\infty. \end{aligned}$$

The *No Free Lunch with Vanishing Risk* (NFLVR) condition is defined as: $A_3 \cap L_+^0 = \{0\}$.

The traditional FTAP (see [15], [28]) states that a model satisfies the NFLVR condition if and only if there exists an equivalent *sigma-martingale measure*, i.e. a measure $\mathsf{Q} \sim \mathsf{P}$ such that S is an $(\mathcal{F}_t, \mathsf{Q})$-sigma-martingale. Recall that a process $(X_t)_{t \in [0,T]}$ is called a *sigma-martingale* if there exists a sequence of predictable sets $(D_n)_{n \in \mathbb{N}}$ such that

	Traditional approach	**Proposed approach**
The price process	$\mathbb{R}^d$-valued semimartingale	Infinite-dimensional process with adapted, càdlàg, and bounded below components
Trading strategies	Predictable strategies satisfying the integrability and the admissibility conditions	Simple strategies with no integrability and no admissibility conditions imposed
The variant of the No Arbitrage condition	NFLVR	NGA
FTAP	NFLVR $\Longleftrightarrow$ existence of an equivalent sigma-martingale measure	NGA $\Longleftrightarrow$ existence of an equivalent martingale measure
Set of fair prices of a contingent claim	$(V_*(F), V^*(F))$	$I(F)$

`Table 1.` The differences between the traditional approach to asset pricing in the continuous-time setting and the proposed approach

$D_n \subseteq D_{n+1}$, $\bigcup_n D_n = \Omega \times [0,T]$, and, for any n, the stochastic integral $\int_0^\cdot I_{D_n}(s)dX_s$ is a uniformly integrable martingale (this definition was proposed by Goll and Kallsen [21]; it is equivalent to the original definition of Chou [8] and Émery [17]). The class of sigma-martingales contains the class of local martingales and is wider as shown by the Émery example (see [17]). However, an $\mathbb{R}^d_+$-valued sigma-martingale is necessarily a local martingale as shown by Ansel and Stricker [1].

The set of fair prices of a contingent claim F is defined as the interval with the endpoints $V_*(F)$ and $V^*(F)$, where

$$\begin{aligned} V_*(F) &= \sup\{x : \text{there exists } X \in A \text{ such that } x - X \le F \ \mathsf{P}\text{-a.s.}\},\\ V^*(F) &= \inf\{x : \text{there exists } X \in A \text{ such that } x + X \ge F \ \mathsf{P}\text{-a.s.}\} \end{aligned}$$

(here A is given by (4.2)). It follows from [15], [18], and [19] that if the NFLVR condition is satisfied and F is bounded below, then

$$V^*(F) = \sup_{\mathsf{Q} \in \mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F, \tag{4.3}$$

where

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-sigma-martingale}\}. \tag{4.4}$$

Let us now give 4 examples and 2 remarks, which illustrate the problems that arise when one applies the traditional approach.

The first two examples and the remark following them show that the admissibility condition leads to an inadmissible restriction of the class of strategies (by a strategy we mean a process H that appears in (4.2)).

Example 4.3. Consider the Black–Scholes model, i.e. $S_t = e^{\mu t+\sigma B_t}$, where B is a Brownian motion. Let $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. Then the strategy $H = -1$ is not admissible. In other words, the admissibility condition prohibits in this model the strategy that consists in the short selling of the asset at time 0 and buying it back at time T. □

Example 4.4. Consider the exponential Lévy model, i.e. $S_t = e^{X_t}$, where X is a Lévy process. Let $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. Suppose that the jumps of X are not bounded from above (the majority of the exponential Lévy models used in the modern financial mathematics satisfy this condition). One can check that if H is an admissible strategy, then $H(\omega, t) \geq 0$ $\mathsf{P} \times \mu_L$-a.e, where μ_L is the Lebesgue measure on $[0, T]$. In other words, the admissibility condition prohibits in this model all the strategies employing short selling. Clearly, this is an unacceptable restriction: for example, when hedging a put option in practice, one employs strategies H with $H < 0$ (for more details, see [24; Ch. 14]). □

`Remark`. Another drawback of the admissibility condition is as follows. Such a condition is not imposed in the discrete-time models, but it is imposed in the continuous-time models. This leads to an unpleasant unbalance. In particular, when one embeds a discrete-time model into a continuous-time model, then the set of attainable incomes defined for this continuous-time model by (4.2) does not coincide with the set of attainable incomes defined for the original discrete-time model.

The next example shows that in some models the traditional interval of fair prices is too wide.

Example 4.5. Let $S_t = I(t < T) + \xi I(t = T)$, where ξ is an $\mathbb{R}_+$-valued random variable with the property: for any $a > 0$, $\mathsf{P}(\xi < a) > 0$ and $\mathsf{P}(\xi > a) > 0$. Let $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. Consider $F = S_T$.

Let us find $V_*(F)$. Let H be a predictable admissible strategy and $x \in \mathbb{R}$ be such that

$$x - \int_0^T H_u dS_u \leq F. \tag{4.5}$$

Note that

$$\int_0^T H_u dS_u = H_T \Delta S_T = H_T(\xi - 1).$$

Since H is $(\mathcal{F}_t)$-predictable and $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t < T$, H_T is a real number. The admissibility condition, together with the property $\mathsf{P}(\xi > a) > 0$ for any $a > 0$, shows that $H_T \geq 0$. This, combined with (4.5) and with the property $\mathsf{P}(\xi < a) > 0$ for any $a > 0$, yields $x \leq 0$. Consequently, $V_*(F) = 0$.

In a similar way one checks that $V^*(F) = 1$. Thus, the interval of fair prices provided by the traditional approach is $[0, 1]$. On the other hand, the interval of fair prices provided by common sense consists only of point 1 since F can be replicated by buying the asset (whose discounted price is S) at time 0. □

Remark. In the model of the previous example, we have, due to the result of Ansel and Stricker [1],

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-local martingale}\}$$

($\mathcal{M}_\sigma$ is given by (4.4)). Furthermore, for any $(\mathcal{F}_t)$-stopping time τ, we have either $\tau = T$ P-a.s. or $\tau < T$ P-a.s. Consequently,

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\} = \{\mathsf{Q} \sim \mathsf{P} : \mathsf{E}_\mathsf{Q}\xi = 1\}.$$

Therefore, $\inf_{\mathsf{Q}\in\mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F = 1$. This shows that the equality $V_*(F) = \inf_{\mathsf{Q}\in\mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F$, which is dual to (4.3), is not true for F bounded below.

One way to overcome this problem was proposed in [15]. Namely, the authors of that paper altered the definition of $V_*(F)$ and $V^*(F)$ by introducing the so-called w-admissibility condition as a substitute for the admissibility condition. However, a weak point of this definition is that it depends on the choice of a so-called weight function.

The fourth example is the most striking one. It shows that the use of the traditional approach may lead to mispricing contingent claims.

Example 4.6. Let $S_t = |B_t|^{-1}$, where B is a 3-dimensional Brownian motion started at a point $B_0 \neq 0$. Let $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. Without loss of generality, $B_0^2 = B_0^3 = 0$. Note that

$$\mathsf{E}S_T = \mathsf{E}((B_T^1)^2 + (B_T^2)^2 + (B_T^3)^2)^{-1/2} \le \mathsf{E}((B_T^2)^2 + (B_T^3)^2)^{-1/2} = \frac{const}{\sqrt{T}}.$$

We take T large enough, so that $\mathsf{E}S_T < S_0$ (actually, $\mathsf{E}S_T < S_0$ for any $T > 0$). Consider $F = S_T$.

Let us find $V^*(F)$. Applying Itô's formula and P. Lévy's characterization theorem (see [36; Ch. IV, Th. 3.6]), we conclude that

$$S_t = S_0 + \int_0^t S_u^2 dW_u, \quad t \in [0, T], \tag{4.6}$$

where W is an $(\mathcal{F}_t, \mathsf{P})$-Brownian motion. Furthermore, Itô's theorem (see [34; Th. 5.2.1]) guarantees that (S, W) is a strong solution of stochastic differential equation (4.6), i.e. $\mathcal{F}_t^S \subseteq \mathcal{F}_t^W$. It is clear from (4.6) that $\mathcal{F}_t^W \subseteq \mathcal{F}_t^S$, and hence, $\mathcal{F}_t^W = \mathcal{F}_t^S = \mathcal{F}_t$. Set $F_t = \mathsf{E}(F \,|\, \mathcal{F}_t)$. By the representation theorem for the Brownian motion (see [36; Ch. V, Th. 3.5]), there exists an $(\mathcal{F}_t)$-predictable W-integrable process K such that

$$F_t = \mathsf{E}F + \int_0^t K_u dW_u, \quad t \in [0, T].$$

In view of (4.6),

$$F_t = \mathsf{E}F + \int_0^t \frac{K_u}{S_u^2} dS_u = \mathsf{E}F + \int_0^t H_u dS_u, \quad t \in [0, T]. \tag{4.7}$$

Since $F_t \ge 0$, the strategy H is admissible. Consequently, $V^*(F) \le \mathsf{E}F$.

Similarly, by considering $F_t^n = \mathsf{E}(FI(F \le n) \,|\, \mathcal{F}_t)$, we prove that $V_*(F) \ge \mathsf{E}F$. As a result, the fair price provided by the traditional approach is $\mathsf{E}F = \mathsf{E}S_T$. On the other hand, the fair price provided by common sense is S_0, which is not equal to $\mathsf{E}S_T$! □

The problems described above do not arise in the approach proposed in this paper.

Indeed, no admissibility restriction is imposed in this approach, which solves the problems described in Examples 4.3, 4.4, and the remark following Example 4.4.

In Example 4.5, we have, due to Theorem 3.10,

$$I(F) = \{\mathsf{E}_{\mathsf{Q}}F : \mathsf{Q} \in \mathcal{M}\} = \{\mathsf{E}_{\mathsf{Q}}F : \mathsf{Q} \sim \mathsf{P},\ \mathsf{E}_{\mathsf{Q}}\xi = 1\} = \{1\},$$

which agrees with common sense.

By Theorem 3.10, the left endpoint of $I(F)$ coincides with $\inf_{\mathsf{Q}\in\mathcal{M}} \mathsf{E}_{\mathsf{Q}}F$ for any F bounded below, which solves the problem mentioned in the remark following Example 4.5.

Finally, in Example 4.6, P is the only local martingale measure for S. Indeed, if $\mathsf{Q} \sim \mathsf{P}$ is a local martingale measure for S, then S satisfies equation (4.6) with respect to Q. By Itô's theorem (see [34; Th. 5.2.1]), there are strong existence and pathwise uniqueness for this equation, and the Yamada–Watanabe theorem (see [36; Ch. IX, Th. 1.7]) implies the uniqueness in law. Hence, $\mathsf{Q} = \mathsf{P}$. Since P is not a martingale measure, there exists no equivalent martingale measure. This means that the model considered in Example 4.6 does not satisfy the NGA condition, and the paradox is solved.

`Remark`. An "arbitrage opportunity" in the model of Example 4.6 can be constructed as follows. Consider the strategy $G = H - 1$, where H is given by (4.7). Then

$$\int_0^T G_u dS_u = \int_0^T H_u dS_u - S_T + S_0 = -\mathsf{E}_{\mathsf{P}} S_T + S_0 > 0.$$

The strategy G is not admissible, so it does not yield a free lunch with vanishing risk opportunity. It does not yield a generalized arbitrage opportunity either, but it can be used to construct a generalized arbitrage opportunity as follows. There exist simple strategies $(\widetilde{H}_n)_{n\in\mathbb{N}}$ such that

$$\sup_{t\in[0,T]} \left| \int_0^t \widetilde{H}_{nu} dS_u - \int_0^t H_u dS_u \right| \xrightarrow[n\to\infty]{\mathsf{P}} 0.$$

Set

$$\tau_n = \inf\left\{ t \in [0,T] : \int_0^t \widetilde{H}_{nu} dS_u \le -\mathsf{E}_{\mathsf{P}} S_T - 1 \right\},$$
$$H_{nt} = \widetilde{H}_{nt} I(t \le \tau_n), \quad t \in [0,T].$$

Since

$$\int_0^t H_u dS_u \ge -\mathsf{E}_{\mathsf{P}} S_T, \quad t \in [0,T],$$

we get

$$\int_0^T H_{nu} dS_u \xrightarrow[n\to\infty]{\mathsf{P}} \int_0^T H_u dS_u = S_T - \mathsf{E}_{\mathsf{P}} S_T.$$

Set $G_n = H_n - 1$. Then, for $X_n = \int_0^T G_{nu} dS_u$, we have $X_n \in A$, where A is given by (4.1). Furthermore, $X_n \ge -S_T + S_0 - \mathsf{E}_{\mathsf{P}} S_T - 1$ P-a.s. for any $n \in \mathbb{N}$ and

$$X_n \xrightarrow[n\to\infty]{\mathsf{P}} S_0 - \mathsf{E}_{\mathsf{P}} S_T > 0.$$

Note that $Z := S_T - S_0$ belongs to B, where B is given by (3.1). Take $Y_n \in L^0_+$, $n \in \mathbb{N}$ such that

$$\frac{X_n - Y_n}{Z + \gamma(Z)} = \frac{X_n}{Z + \gamma(Z)} \wedge (S_0 - \mathsf{E_P} S_T),$$

and then

$$\frac{X_n - Y_n}{Z + \gamma(Z)} \xrightarrow[n\to\infty]{\sigma(L^\infty, L^1(\mathsf{P}))} \frac{S_0 - \mathsf{E_P} S_T}{Z + \gamma(Z)}.$$

This yields a generalized arbitrage opportunity in the model of Example 4.6.

One of the problems associated with the model under consideration is related to the *change of numéraire*. It is as follows. Let $S = (S^1_t, \dots, S^d_t)_{t\in[0,T]}$ be the price process of d assets. We assume that each of its components is strictly positive. Fix $\alpha^1, \dots, \alpha^d \ge 0$ with $\sum_{i=1}^d \alpha^i > 0$ and define a numéraire as the combination $\sum_{i=1}^d \alpha^i S^i$. Now, define the discounted price process as $\overline{S} = S/\sum_{i=1}^d \alpha^i S^i$ and define the set of attainable $\overline{A}$ incomes by (4.1) or (4.2), depending on the choice of the approach. Now, let us choose another combination $\sum_{i=1}^d \beta^i S^i$ as a numéraire, define the new discounted process $\widetilde{S}$ as $\widetilde{S} = S/\sum_{i=1}^d \beta^i S^i$ and define the set of attainable incomes $\widetilde{A}$ through $\widetilde{S}$. The problem is whether the NFLVR/NGA property holds or does not hold for the models $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ and $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ simultaneously.

In the traditional approach, the answer is negative as shown by the example below (it is borrowed from [13]). Let us mention in this connection the papers [14] and [16] devoted to the study of conditions under which the NFLVR property is preserved under the change of numéraire.

Example 4.7. Let $S^0 = 1$ and $S^1 = |B|^{-1}$, where B is a 3-dimensional Brownian motion started at a point $B_0 \neq 0$. Let $\mathcal{F}_t = \mathcal{F}^S_t$, $\mathcal{F} = \mathcal{F}_T$. If we take $\overline{S} = S/S^0$, then the model $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ ($\overline{A}$ is defined by (4.2)) satisfies the NFLVR condition since the process S^1 is a local martingale with respect to the original probability measure (see representation (4.6)). On the other hand, if we take $\widetilde{S} = S/S^1$, then $\widetilde{S}^0 = |B|$ (this is a 3-dimensional Bessel process). If Q is an equivalent sigma-martingale measure for $\widetilde{S}$, then, by the result of Ansel and Stricker [1], $\widetilde{S}^0$ is an $(\mathcal{F}_t, \mathsf{Q})$-local martingale. Using Itô's formula, one easily checks that the quadratic variation of $\widetilde{S}^0$ is given by $[\widetilde{S}^0]_t = t$. P. Lévy's characterization theorem (see [36; Ch. IV, Th. 3.8]) now implies that $\widetilde{S}^0$ is a Q-Brownian motion. But this contradicts the positivity of $\widetilde{S}^0$. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ does not satisfy the NFLVR condition. □

In contrast, the change of numéraire preserves the NGA property as shown by the statement below.

Theorem 4.8 (Change of numéraire). *Let $\overline{A}$ (resp., $\widetilde{A}$) be defined through $\overline{S}$ (resp., $\widetilde{S}$) by (4.1). Then the models $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ and $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ satisfy or do not satisfy the NGA condition simultaneously.*

The proof employs the following statement (see [27; Ch. III, Prop. 3.8]).

Lemma 4.9. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space and $\mathsf{Q} \ll \mathsf{P}$. Let $Z_t = \frac{d\mathsf{Q}|\mathcal{F}_t}{d\mathsf{P}|\mathcal{F}_t}$ be the density process of Q with respect to P. Then a process M is an $(\mathcal{F}_t, \mathsf{Q})$-martingale if and only if MZ is an $(\mathcal{F}_t, \mathsf{P})$-martingale.*

`Proof of Theorem 4.8.` Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ satisfies the NGA condition. Then there exists a probability measure $\overline{\mathsf{Q}} \sim \mathsf{P}$ such that $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale. Let $\overline{Z}$ denote the density process of $\overline{\mathsf{Q}}$ with respect to P. Consider the process

$$\widetilde{Z} = c\overline{Z} \frac{\sum_{i=1}^{d} \beta^i S^i}{\sum_{i=1}^{d} \alpha^i S^i},$$

where the constant c is chosen in such a way that $\widetilde{Z}_0 = 1$. As $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale, then, by Lemma 4.9, $\widetilde{Z}$ is an $(\mathcal{F}_t, \mathsf{P})$-martingale. Hence, $\widetilde{Z}$ is the density process of the probability measure $\widetilde{\mathsf{Q}} = \widetilde{Z}_T \mathsf{P}$ with respect to P (note that $\widetilde{\mathsf{Q}} \sim \mathsf{P}$ since $\widetilde{Z}$ is strictly positive). As $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale, then, by Lemma 4.9, the process $\widetilde{S}\widetilde{Z} = c\overline{S}\,\overline{Z}$ is an $(\mathcal{F}_t, \mathsf{P})$-martingale, which (again by Lemma 4.9) implies that $\widetilde{S}$ is an $(\mathcal{F}_t, \widetilde{\mathsf{Q}})$-martingale. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ satisfies the NGA condition. □

5 Dynamic Model with Infinite Time Horizon

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t^i)_{t \in \mathbb{R}_+}$, $i \in I$ be a family of real-valued $(\mathcal{F}_t)$-adapted càdlàg processes with components bounded below. From the financial point of view, S_t^i is the discounted price of the i-th asset at time t. Define the set of attainable incomes by

$$\begin{aligned} A = \Bigg\{ &\sum_{n=1}^{N} \sum_{i \in I} H_n^i (S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \le \dots \le u_N \text{ are } (\mathcal{F}_t)\text{-stopping times} \\ &\text{such that } \{u_N = \infty\} \subseteq \big\{\text{for any } i,\ \exists \lim_{t \to \infty} S_t^i\big\},\ H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable,} \\ &\text{and } H_n^i = 0 \text{ for all } i,\ \text{except for a finite set} \Bigg\}. \end{aligned} \tag{5.1}$$

`Notation`. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for any } i \in I,\ S^i \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-uniformly integrable martingale}\}.$$

Key Lemma 5.1. *Suppose that I is countable and, for any i, the limit $S_\infty^i = \lim_{t \to \infty} S_t^i$ exists P-a.s. Then, for the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}\bigg(\sum_{i \in I} \lambda^i (S_\infty^i - S_0^i)\bigg) = \mathcal{M},$$

where constants $\lambda^i > 0$ are chosen in such a way that $\sum_{i \in I} \lambda^i S_T^i < \infty$ a.s. and $\sum_{i \in I} \lambda^i S_0^i < \infty$.

`Proof`. Note that $(S_t^i)_{t \in \mathbb{R}_+}$ is an $(\mathcal{F}_t, \mathsf{Q})$-uniformly integrable martingale if and only if $(S_t^i)_{t \in [0,\infty]}$ is a $(\mathcal{G}_t, \mathsf{Q})$-martingale, where

$$\mathcal{G}_t = \begin{cases} \mathcal{F}_t & \text{if } t \in \mathbb{R}_+, \\ \mathcal{F} & \text{if } t = \infty \end{cases}$$

(this statement follows from [36; Ch. II, Th. 3.1]). The desired statement can now be proved in the same way as Key Lemma 4.1. □

Since Key Lemma 5.1 contains an additional assumption, Theorem 3.6 cannot be applied immediately, and the proof of the FTAP in this model requires a bit of additional work.

Corollary 5.2. *Suppose that I is countable. Then the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition if and only if there exists an equivalent uniformly integrable martingale measure (i.e. $\mathcal{M} \neq \emptyset$).*

Proof. *Step 1.* Let us prove the "only if" implication. Lemma 3.7 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A_4(0)$ yields a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X \le 0$ for any $X \in A$ that is bounded below. For any $i \in I$, any $u \le v \in \mathbb{R}_+$, and any $D \in \mathcal{F}_u$ such that S^i_u is bounded on D, the random variable $I_D(S^i_v - S^i_u)$ is bounded below, and hence, $\mathsf{E}_\mathsf{Q} I_D(S^i_v - S^i_u) \le 0$. This shows that S^i is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale. By Doob's supermartingale convergence theorem (see [36; Ch. II, Th. 2.10]), the limit $\lim_{t\to\infty} S^i_t$ exists Q-a.s., and hence, P-a.s. Now, Theorem 3.6, combined with Key Lemma 5.1, yields the desired statement.

Step 2. Let us prove the "if" implication. Take $\mathsf{Q} \in \mathcal{M}$. Then, by Doob's theorem, for any $i \in I$, $\lim_{t\to\infty} S^i_t$ exists Q-a.s., and hence, P-a.s. Now, Theorem 3.6, combined with Key Lemma 5.1, yields the desired statement. □

It has been shown in the proof of Corollary 5.2 that the NGA condition implies the existence of $\lim_{t\to\infty} S_t$ P-a.s. Hence, Theorem 3.10 can be applied with no additional assumptions.

It would be more natural to define the set of attainable incomes in this model as

$$A = \Bigg\{ \sum_{n=1}^N \sum_{i\in I} H^i_n (S^i_{u_n} - S^i_{u_{n-1}}) : N \in \mathbb{N},\ u_0 \le \dots \le u_N < \infty \text{ are } (\mathcal{F}_t)\text{-stopping times,} \\ H^i_n \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H^i_n = 0 \text{ for all } i, \text{ except for a finite set} \Bigg\}.$$

However, for this choice of A we can only establish the equality $\mathcal{R} = \mathcal{M}$ (in the lemma below, I is arbitrary), but we cannot prove that Assumption 3.5 is satisfied.

Lemma 5.3. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have $\mathcal{R} = \mathcal{M}$.*

Proof. *Step 1.* The inclusion $\mathcal{M} \subseteq \mathcal{R}$ follows from the similar inclusion in Key Lemma 5.1.

Step 2. Let us prove the inclusion $\mathcal{R} \subseteq \mathcal{M}$. Choose $\mathsf{Q} \in \mathcal{R}$. Fix $i \in I$. For any $u \le v \in \mathbb{R}_+$ and $D \in \mathcal{F}_u$, we have $\mathsf{E}_\mathsf{Q} I_D(S^i_v - S^i_u) = 0$ since S^i is bounded below. Hence, S^i is an $(\mathcal{F}_t, \mathsf{Q})$-martingale.

By Doob's supermartingale convergence theorem, there exists a limit $S^i_\infty = (\text{a.s.}) \lim_{t\to\infty} S^i_t$. By the Fatou lemma for conditional expectations,

$$\mathsf{E}_\mathsf{Q}(S^i_\infty \mid \mathcal{F}_t) \le S^i_t, \quad t \ge 0. \tag{5.2}$$

In particular, $\mathsf{E}_\mathsf{Q} S^i_\infty \le S^i_0$.

Suppose that $\mathsf{E}_\mathsf{Q} S^i_\infty < S^i_0$. The process $X_t = \mathsf{E}_\mathsf{Q}(S^i_\infty \mid \mathcal{F}_t)$, $t \ge 0$ has a càdlàg modification. Furthermore, $X_t \xrightarrow[t\to\infty]{\mathsf{Q}\text{-a.s.}} S^i_\infty$. Consequently, the stopping time

$$\tau = \inf\left\{ t \ge 0 : |S^i_t - X_t| \le \frac{S^i_0 - \mathsf{E}_\mathsf{Q} S^i_\infty}{2} \right\}$$

is finite Q-a.s. It follows from the inclusion $\mathsf{Q} \in \mathcal{R}$ and the positivity of S^i that $\mathsf{E}_\mathsf{Q} S^i_\tau = S^i_0$. Thus,

$$\mathsf{E}_\mathsf{Q} X_\tau \ge S^i_0 - \frac{S^i_0 - \mathsf{E}_\mathsf{Q} S^i_\infty}{2} > \mathsf{E}_\mathsf{Q} S^i_\infty.$$

But this contradicts the equality $\mathsf{E}_\mathsf{Q} X_\tau = \mathsf{E}_\mathsf{Q} S^i_\infty$, which is a consequence of the optional stopping theorem for uniformly integrable martingales (see [36; Ch. II, Th. 3.2]). As a result, $\mathsf{E}_\mathsf{Q} S^i_\infty = S^i_0$. This, combined with (5.2), yields $\mathsf{E}_\mathsf{Q}(S^i_\infty \mid \mathcal{F}_t) = S^i_t$, $t \ge 0$. The proof is completed. □

The traditional approach to the arbitrage pricing in dynamic models with the infinite time horizon is the same as the one for continuous-time models with a finite time horizon. The only difference is that the set of attainable incomes given by (4.2) should be replaced by

$$A = \bigg\{ \int_0^\infty H_u dS_u : H \text{ is } (\mathcal{F}_t)\text{-predictable, } S\text{-integrable,}$$
$$\text{admissible, and such that } \lim_{t\to\infty} \int_0^t H_u dS_u \text{ exists } \mathsf{P}\text{-a.s.} \bigg\}.$$

Here $\int_0^\infty H_u dS_u := \lim_{t\to\infty} \int_0^t H_u dS_u$. (This might be called the improper stochastic integral. Alternatively, one can use the stochastic integral up to infinity; see [7]. The FTAP remains the same for these two types of integrals.)

Many models with the infinite time horizon that are arbitrage-free in the traditional approach (i.e. satisfy the NFLVR condition for predictable admissible strategies) are not arbitrage-free in the proposed approach (i.e. do not satisfy the NGA condition for simple strategies). This is illustrated by the following example.

Example 5.4. Let $S_t = e^{B_t - t/2}$, where B is a Brownian motion. Let $\mathcal{F}_t = \mathcal{F}^S_t$, $\mathcal{F} = \bigvee_{t\ge 0} \mathcal{F}_t$. This model satisfies the NFLVR condition since the process S is a martingale (and hence, a sigma-martingale) with respect to the original probability measure. On the other hand, this model does not satisfy the NGA condition. Indeed, consider the stopping time $v = \inf\{t \ge 0 : S_t = 1/2\}$. Then the random variable $-S_v + S_0 = 1/2$ belongs to the set A given by (5.1). Hence, the NGA condition is violated.

From the financial point of view, the strategy providing generalized arbitrage in this model consists in the short selling of the asset at time 0 and buying it back at time v. Note that this strategy is prohibited in the traditional approach by the admissibility condition. □

`Remark.` A *"buy and hold" strategy* consists in buying an asset, waiting until its discounted price reaches some higher level, and selling it back at that time. The opposite (it may be called "sell and wait") strategy consists in the short selling of an asset, waiting until its discounted price reaches some lower level and buying it back at that time. In many models (like the one described above) such "sell and wait" strategies lead to arbitrage opportunities. In the traditional approach, these strategies are prohibited by the admissibility condition. In the approach proposed here, such strategies are allowed, but the models, in which they yield arbitrage opportunities, are "prohibited" in the sense that they do not satisfy the NGA condition. Indeed, if the NGA condition is satisfied, then there exists an equivalent uniformly integrable martingale measure. But a uniformly integrable martingale with a strictly positive probability never reaches a preassigned level, so that in models satisfying the NGA condition the "sell and wait" strategy does not yield an arbitrage opportunity.

To conclude the section, we show that no "stationary" model with the infinite time horizon satisfies the NGA condition. We say that a real-valued process Z has *stationary increments* if $Z_{t+h} - Z_{s+h} \overset{\text{Law}}{=} Z_t - Z_s$ for any $s \le t \in \mathbb{R}_+$, $h \in \mathbb{R}_+$.

Proposition 5.5. *Let $S_t = S_0 e^{Z_t}$ where Z has stationary increments and $\mathsf{P}(Z_t \neq Z_0) > 0$ for some $t \in \mathbb{R}_+$. Then the NGA condition is not satisfied.*

`Proof`. Suppose that the NGA condition is satisfied. Without loss of generality, we can assume that $\mathsf{P}(Z_t \neq Z_0) > 0$ for some $t \in \mathbb{R}_+$. The reasoning used in the proof of Corollary 5.2 shows that there exists $\lim_{t\to\infty} S_t$ P-a.s. Hence, there exists $\lim_{t\to\infty} Z_t =: Z_\infty$ P-a.s. (this limit takes on values in $[-\infty, \infty)$). Denote $\mathsf{P}(Z_\infty > -\infty)$ by p. Fix $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that $N > 1/\varepsilon$ and

$$\mathsf{P}(Z_\infty > -\infty \text{ and } |Z_n - Z_\infty| < \varepsilon \text{ for any } n \ge N) > p - \varepsilon.$$

Then

$$\mathsf{P}(Z_\infty > -\infty \text{ and } |Z_{2N} - Z_N| < 2\varepsilon) > p - \varepsilon.$$

Since $Z_{2N} - Z_N \overset{\text{Law}}{=} Z_N$, we get $\mathsf{P}(|Z_N| < 2\varepsilon) > p - \varepsilon$. As ε can be chosen arbitrarily small, we conclude that $\mathsf{P}(Z_\infty = 0) = p$. Hence, $Z_\infty = 0$ P-a.e. on the set $\{Z_\infty > -\infty\}$. This means that Z_∞ takes on only values $-\infty$ and 0.

Take $t \in \mathbb{R}_+$ such that $\mathsf{P}(Z_t \neq Z_0) > 0$. Choose $\alpha > 0$ such that $\mathsf{P}(|Z_t - Z_0| > \alpha) > 0$. For any $T \in \mathbb{R}_+$,

$$\mathsf{P}(|Z_{T+t} - Z_T| > \alpha) = \mathsf{P}(|Z_t - Z_0| > \alpha) > 0.$$

Consequently, $\mathsf{P}(Z_\infty = 0) < 1$. Thus, S_∞ takes on only values 0 and S_0, and $\mathsf{P}(S_\infty = 0) > 0$. Then $\mathcal{M} = \emptyset$, and, by Corollary 5.2, the NGA condition is not satisfied. □

Corollary 5.6. *Let $S_t = S_0 e^{Z_t}$ where Z is a Lévy process that is not identically equal to zero. Then the NGA condition is not satisfied.*

6 Model with European Call Options as Basic Assets

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $T \in [0, \infty)$. Let S_T be an $\mathbb{R}_+$-valued random variable. From the financial point of view, S_T is the price of some asset at time T. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices of European call options on this asset with maturity T (in practice $\mathbb{K}$ is finite, but in theory it is often assumed that $\mathbb{K} = \mathbb{R}_+$) and let $\varphi(K)$, $K \in \mathbb{K}$ be the price at time 0 of a European call option with the payoff $(S_T - K)^+$. Define the set of attainable incomes by

$$A = \left\{ \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) : N \in \mathbb{N},\ K_n \in \mathbb{K},\ h_n \in \mathbb{R} \right\}.$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading at times 0 and T European call options on "our" asset with maturity T. We assume that $0 \in \mathbb{K}$, which means the possibility to trade the underlying asset.

`Notation`. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \operatorname{Law}_{\mathsf{Q}} S_T \in \mathcal{D}\},$$

where

$$\begin{aligned}\mathcal{D} = \{\psi'' : \psi \text{ is convex on } \mathbb{R}_+,\ \psi'_+(0) \geq -1,\ \lim_{x\to\infty} \psi(x) = 0,\\ \text{and } \psi(K) = \varphi(K),\ K \in \mathbb{K}\}.\end{aligned}$$

Here ψ'_+ denotes the right-hand derivative and ψ'' denotes the second derivative taken in the sense of distributions (i.e. $\psi''((a, b]) = \psi'_+(b) - \psi'_+(a)$) with the convention: $\psi''(\{0\}) = \psi'_+(0) + 1$ (thus, ψ'' is a probability measure provided that $\psi'_+(0) \geq -1$).

Key Lemma 6.1. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$, *we have*

$$\mathcal{R} = \mathcal{R}(S_T - \varphi(0)) = \mathcal{M}.$$

`Proof`. *Step 1.* The inclusion $\mathcal{R} \subseteq \mathcal{R}(S_T - \varphi(0))$ follows from Lemma 3.4.

Step 2. Let us prove the inclusion $\mathcal{R}(S_T - \varphi(0)) \subseteq \mathcal{M}$. Fix $\mathsf{Q} \in \mathcal{R}(S_T - \varphi(0))$. By considering the function $\psi(x) = \mathsf{E}_{\mathsf{Q}}(S_T - x)^+$, we conclude that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Fix $\mathsf{Q} \in \mathcal{M}$. Then

$$\mathsf{E}_{\mathsf{Q}}(S_T - K)^+ = \int_{\mathbb{R}_+} (x - K)^+ \psi''(dx) = \psi(K) = \varphi(K), \quad K \in \mathbb{K}.$$

Consequently, $\mathsf{E}_{\mathsf{Q}} X = 0$ for any $X \in A$, which implies that $\mathsf{Q} \in \mathcal{R}$. $\square$

Recalling Theorems 3.6 and 3.10, we get

Corollary 6.2. *Let* $\mathbb{K} = \mathbb{R}_+$.

(i) *The NGA is satisfied if and only if*

(a) φ *is convex*;
(b) $\varphi'_+(0) \geq -1$;
(c) $\lim_{x\to\infty} \varphi(x) = 0$;
(d) $\varphi'' \sim \operatorname{Law}_{\mathsf{P}} S_T$.

(ii) *Suppose that the NGA is satisfied. Let* $F = f(S_T)$, *where* f *is bounded below. Then*

$$I(F) = \begin{cases} \left\{\int_{\mathbb{R}_+} f(x)\varphi''(dx)\right\} & \text{if } \int_{\mathbb{R}_+} f(x)\varphi''(dx) < \infty,\\ \emptyset & \text{otherwise.}\end{cases}$$

We conclude this section by three interesting examples. The first example shows that the ordinary NA condition (which means that $A \cap L^0_+ = \{0\}$) is too weak for the model under consideration.

Example 6.3. Let $\mathbb{K} = \mathbb{R}_+$,

$$\mathsf{P}(S_T \in A) = \frac{1}{2}\left(I(1 \in A) + \int_A e^{-x} dx\right), \quad A \in \mathcal{B}(\mathbb{R}_+),$$

and $\varphi(K) = e^{-K}$. This model satisfies the NA condition. Indeed, suppose that there exists

$$X = \sum_{n=1}^N h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

such that $X \geq 0$ P-a.s. and $\mathsf{P}(X > 0) > 0$. Note that X can be represented as $X = f(S_T)$ with a continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}_+} f(x)e^{-x} dx = \sum_{n=1}^N h_n \int_{\mathbb{R}_+} ((x - K_n)^+ - e^{-K_n})e^{-x} dx = 0. \tag{6.1}$$

The above assumptions on X imply that $f \geq 0$ everywhere and f is not identically equal to zero. But this contradicts (6.1). Thus, the NA condition is satisfied.

Consider now $F = I(S_T = 1)$. For every $\varepsilon > 0$, consider the function $f_\varepsilon(x) = (1 - \varepsilon^{-1}|x - 1|)^+$. It is seen from the representation

$$f_\varepsilon(S_T) = \frac{1}{\varepsilon}(S_T - 1 + \varepsilon)^+ - \frac{2}{\varepsilon}(S_T - 1)^+ + \frac{1}{\varepsilon}(S_T - 1 - \varepsilon)^+$$

that the random variables

$$X_\varepsilon = f_\varepsilon(S_T) - \int_{\mathbb{R}_+} f_\varepsilon(x)e^{-x} dx$$

belong to A and

$$X_\varepsilon + \int_{\mathbb{R}_+} f_\varepsilon(x)e^{-x} dx \geq F.$$

As $\int_{\mathbb{R}_+} f_\varepsilon(x)e^{-x} dx \underset{\varepsilon \downarrow 0}{\longrightarrow} 0$, it is reasonable to conclude that the fair price of F should not exceed 0 (thus, the fair price should equal 0 since F is positive). But on the other hand, $\mathsf{P}(F = 1) = 1/2$, so that we obtain a contradiction with common sense. The reason is that this model is not "fair" because one can construct "asymptotic arbitrage" taking X_ε with $\varepsilon \downarrow 0$. □

The second example shows that the NFL condition (see Remark (iii) following Definition 3.2) is also too weak for the model under consideration.

Example 6.4. Let $\mathbb{K} = \mathbb{R}_+$, $\mathsf{P}(S_T \leq x) = 1 - e^{-x}$, and $\varphi(K) = e^{-K} + 1$. This model satisfies the NFL condition. Indeed, let

$$X = \sum_{n=1}^N h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

be bounded below. Note that X can be represented as $X = f(S_T)$ with a continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\lim_{x \to \infty} \frac{f(x)}{x} = \sum_{n=1}^N h_n.$$

The assumption on X implies that $\sum_{n=1}^{N} h_n \geq 0$. Then we can write

$$X \leq \sum_{n=1}^{N} h_n \big((S_T - K_n)^+ - e^{-K_n} \big) = g(S_T) - \int_{\mathbb{R}_+} g(x) e^{-x} dx = g(S_T) - \mathsf{E_P} g(S_T)$$

with $g(x) = \sum_{n=1}^{N} h_n (x - K_n)^+$. This implies that, for any $X \in A_4(0)$ ($A_4(0)$ is defined by (3.2)), we have $\mathsf{E_P} X \leq 0$, so that the NFL condition is satisfied.

On the other hand, in this model the price of a European call option tends to 1 as the strike price tends to $+\infty$, which contradicts common sense. Thus, this model is not "fair" since one can construct "asymptotic arbitrage" by selling European call options with the strike price $K \to +\infty$. □

The third example shows that $I(F)$ might not coincide with the interval, whose end-points are $V_*(F)$ and $V^*(F)$ defined by (3.4) and (3.5). Thus, in general the proposed approach to arbitrage pricing yields a finer interval of fair prices than the traditional approach based on sub- and superreplication.

Example 6.5. Let $\mathbb{K} = \mathbb{R}_+$, $\mathsf{P}(S_T \leq x) = 1 - e^{-x}$, and $\varphi(K) = e^{-K}$. This model satisfies the NGA condition since $\mathsf{P} \in \mathcal{M}$. Choose $D \in \mathcal{B}(\mathbb{R}_+)$ such that, for any $a < b \in \mathbb{R}_+$, the sets $D \cap [a, b]$ and $[a, b] \setminus D$ have a strictly positive Lebesgue measure. Consider $F = I(S_T \in D)$.

Let us find $V^*(F)$ defined by (3.5). Let $x \in \mathbb{R}$ and

$$X = \sum_{n=1}^{N} h_n ((S_T - K_n)^+ - \varphi(K_n)) \in A$$

be such that $x + X \geq F$ P-a.s. We can write

$$X = g(S_T) - \sum_{n=1}^{N} h_n e^{-K_n} = g(S_T) - \sum_{n=1}^{N} h_n \int_{\mathbb{R}_+} (y - K_n)^+ e^{-y} dy = g(S_T) - \int_{\mathbb{R}_+} g(y) e^{-y} dy$$

with $g(y) = \sum_{n=1}^{N} h_n (y - K_n)^+$. Thus,

$$x + g(S_T) - \int_{\mathbb{R}_+} g(y) e^{-y} dy \geq I(S_T \in D) \quad \mathsf{P}\text{-a.s.}$$

Using the continuity of g and the properties of D, we get

$$x + g(z) - \int_{\mathbb{R}_+} g(y) e^{-y} dy \geq 1 \text{ for any } z \in \mathbb{R}_+.$$

This implies that $x \geq 1$. Consequently, $V^*(F) = 1$.

In a similar way one checks that $V_*(F) = 0$. On the other hand, by Corollary 6.2 (ii), $I(F) = \big\{ \int_D e^{-y} dy \big\}$.

7 Mixed Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t)_{t \in [0,T]}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process. From the financial point of view, S_t is the (discounted) price of some asset at time t. Let $\varphi_t(K)$ be the price of a European call option on this asset with maturity t and strike price K (we assume that such an option exists for any $t \in [0,T]$, $K \in \mathbb{R}_+$). Define the set of attainable incomes by

$$\begin{aligned} A = \Bigg\{ &\sum_{m=1}^{M} H_m(S_{u_m} - S_{u_{m-1}}) + \sum_{n=1}^{N} h_n((S_{v_n} - K_n)^+ - \varphi_{v_n}(K_n)) : \\ &M, N \in \mathbb{N},\ u_0 \le \dots \le u_M \text{ are } (\mathcal{F}_t)\text{-stopping times}, \\ &H_n \text{ is } \mathcal{F}_{u_n}\text{-measurable},\ h_n \in \mathbb{R},\ v_n \in [0,T],\ K_n \in \mathbb{R}_+ \Bigg\}. \end{aligned}$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading "our" asset on the interval $[0,T]$ and trading European call options on this asset.

`Notation`. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale and } \operatorname{Law}_{\mathsf{Q}} S_t = \varphi_t'',\ t \in [0,T]\}$$

provided that, for any $t \in [0,T]$, the function φ_t is convex, $(\varphi_t)'_+(0) \ge -1$, and $\lim_{x\to\infty} \varphi_t(x) = 0$. Otherwise, we set $\mathcal{M} = \emptyset$.

Key Lemma 7.1. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$*, we have*

$$\mathcal{R} = \mathcal{R}(S_T - S_0) = \mathcal{M}.$$

`Proof`. *Step 1.* The inclusion $\mathcal{R} \subseteq \mathcal{R}(S_T - S_0)$ follows from Lemma 3.4.

Step 2. Let us prove the inclusion $\mathcal{R}(S_T - S_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(S_T - S_0)$. The proof of Key Lemma 4.1 (Step 2) shows that S is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. For any $t \in [0,T]$, $K \in \mathbb{R}_+$, we have

$$\mathsf{E}_{\mathsf{Q}}(S_t - S_0 - (S_t - K)^+ + \varphi_t(K)) = 0$$

since the random variable under the expectation belongs to A and is bounded. By the martingale property of S, $\mathsf{E}_{\mathsf{Q}}(S_t - S_0) = 0$, which implies that $\mathsf{E}_{\mathsf{Q}}(S_t - K)^+ = \varphi_t(K)$. As a result, $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix

$$X = \sum_{m=1}^{M} H_m(S_{u_m} - S_{u_{m-1}}) + \sum_{n=1}^{N} h_n((S_{v_n} - K_n)^+ - \varphi_{v_n}(K_n)) = X_1 + X_2 \in A.$$

Clearly, X_2 is Q-integrable and $\mathsf{E}_{\mathsf{Q}} X_2 = 0$. The proof of Key Lemma 4.1 (Step 3) shows that $\mathsf{E}_{\mathsf{Q}} X_1^- \ge \mathsf{E}_{\mathsf{Q}} X_1^+$. This leads to the inequality $\mathsf{E}_{\mathsf{Q}} X^- \ge \mathsf{E}_{\mathsf{Q}} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

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